

## ON THE EGOROFF PROPERTY OF POINTWISE CONVERGENT SEQUENCES OF FUNCTIONS

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**ABSTRACT.** The space  $\mathcal{L}(X)$  of real-valued functions on  $X$  has the *Egoroff property* if for any  $\{f_{nk}\}$  such that  $0 \leq f_{nk} \uparrow_k f$  (for every  $n$ ), there exists  $g_m \uparrow f$  such that, for each  $m$  and  $n$ ,  $g_m \leq f_{nk}$  for some  $k$ . We show that  $\mathcal{L}(X)$  has the Egoroff property if and only if the cardinality of  $X$  is smaller than the minimum cardinality of an unbounded family of functions from the set of natural numbers to itself. Therefore, the statement that there is an uncountable set  $X$  such that  $\mathcal{L}(X)$  has the Egoroff property is independent of the axioms of set theory.

The Egoroff property of function spaces, or more generally of vector lattices, is an abstract formulation of Egoroff's theorem from measure theory. Let  $X$  be a set, and let  $\mathcal{L}(X)$  denote the space of all real-valued functions on  $X$ . If  $\{f_k\}$  is a sequence in  $\mathcal{L}(X)$ , then

$$f_k \uparrow_k f$$

means that  $f_0 \leq f_1 \leq \dots \leq f_k \leq \dots$ , and that  $\{f_k\}$  converges pointwise to  $f$ . Following [3, 67.2], we say that  $\mathcal{L}(X)$  has the Egoroff property if the following holds for any doubly indexed sequence  $\{f_{nk}\}$  in  $\mathcal{L}(X)$ :

If  $0 \leq f_{nk} \uparrow_k f$  for all  $n$ , then there exists a sequence  $\{g_m\}$  such that  $g_m \uparrow f$ , and that for each  $m$  and for each  $n$  there is  $k$  such that  $g_m \leq f_{nk}$ .

The Egoroff property of  $\mathcal{L}(X)$  depends only on the cardinality of  $X$ . Clearly,  $\mathcal{L}(X)$  has the property if  $X$  is finite, and an easy diagonal argument shows that  $\mathcal{L}(X)$  has the Egoroff property if  $X$  is countable. It is proved in [3] that, if the continuum hypothesis holds, then  $\mathcal{L}(X)$  has the Egoroff property only if  $X$  is at most countable. In fact, an argument from [1] shows that if  $X$  has the cardinality of the continuum then  $\mathcal{L}(X)$  does not have the Egoroff property.

We show that in the absence of the continuum hypothesis, the space  $\mathcal{L}(X)$  with  $X$  uncountable may or may not have the Egoroff property:

**THEOREM 1.** (a) *It is consistent that  $2^{\aleph_0} > \aleph_1$  and  $\mathcal{L}(X)$  has the Egoroff property only if  $X$  is at most countable.*

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(b) *It is also consistent that  $2^{\aleph_0} > \aleph_1$  and for every  $X$  of size less than  $2^{\aleph_0}$ ,  $\mathcal{L}(X)$  has the Egoroff property.*

Theorem 1 will follow from a more precise result characterizing the cardinalities of the sets  $X$  for which  $\mathcal{L}(X)$  has the Egoroff property. To state this result, we use the following terminology and notation, taken from [4]. A family  $\mathcal{F}$  of functions from the set  $\omega$  of natural numbers into itself is said to be bounded by a function  $g: \omega \rightarrow \omega$  if, for each  $f \in \mathcal{F}$ , the inequality  $f(n) \leq g(n)$  holds for all but finitely many  $n \in \omega$ . If no such  $g$  exists,  $\mathcal{F}$  is unbounded. The minimum possible cardinality for an unbounded family is the bounding number, denoted by  $\mathfrak{b}$ . It is known that  $\mathfrak{b}$  is a regular cardinal, that

$$\aleph_1 \leq \mathfrak{b} \leq 2^{\aleph_0},$$

and that each of the four combinations of equalities and strict inequalities here is consistent with the axioms of set theory. In particular, models obtained by adding many random reals to a model of the continuum hypothesis satisfy  $\mathfrak{b} = \aleph_1$ , while models of Martin's axiom satisfy  $\mathfrak{b} = 2^{\aleph_0}$ ; in both cases,  $2^{\aleph_0}$  can be arbitrarily large [2]. Thus, the following result implies Theorem 1.

**THEOREM 2.**  *$\mathcal{L}(X)$  has the Egoroff property if and only if the cardinality of  $X$  is smaller than  $\mathfrak{b}$ .*

In order to prove Theorem 2, we use a reformulation of the Egoroff property for  $\mathcal{L}(X)$ .

**LEMMA [3, THEOREMS 75.1 AND 73.2].**  *$\mathcal{L}(X)$  has the Egoroff property if and only if the Boolean ring  $P(X)$  does ( $P(X)$  is the power set of  $X$ ), and the Egoroff property for  $P(X)$  is equivalent to the following:*

*Let  $\{A_{nk}\}$  be subsets of  $X$  such that, for all  $n$ ,*

$$(*) \quad A_{n0} \subseteq A_{n1} \subseteq \cdots \subseteq \cdots, \quad \bigcup_{k=0}^{\infty} A_{nk} = A.$$

*Then there exists a sequence  $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n \subseteq \cdots$  such that  $\bigcup_{n=0}^{\infty} B_n = A$ , and for each  $n$ ,  $B_n \subseteq A_{nk}$  for some  $k$ .*

It is this property (\*) that we use in the proof of Theorem 2.

**PROOF OF THEOREM 2.** “*Only if.*” Assume that  $\mathcal{L}(X)$  has the Egoroff property. We shall show that the cardinality of  $X$  is smaller than  $\mathfrak{b}$  by showing that no  $X$ -indexed family of functions  $\omega \rightarrow \omega$  can be unbounded. Let such a family  $\{f_x \mid x \in X\}$  be given, and set

$$A_{nk} = \{x \in X \mid f_x(n) \leq k\}.$$

Clearly, for each fixed  $n$ , the sequence  $\{A_{nk}\}_{k=0}^{\infty}$  is increasing and its union is  $X$ . As  $\mathcal{L}(X)$  has the Egoroff property, the lemma gives us an increasing sequence  $\{B_n\}$ , with union  $X$ , such that for each  $n$  there exists  $k$  with  $B_n \subseteq A_{nk}$ . Let  $g: \omega \rightarrow \omega$  be a function assigning to each  $n$  such a  $k$ ; so  $B_n \subseteq A_{n,g(n)}$ . Consider any fixed  $x \in X$ . As the  $B_n$ 's are increasing and have union  $X$ , we have  $x \in B_n$  for all sufficiently

large  $n$ . Thus, for all sufficiently large  $n$ ,

$$x \in B_n \subseteq A_{n, g(n)},$$

so, by definition of  $A_{nk}$ ,  $f_x(n) \leq g(n)$ . Since  $x \in X$  was arbitrary, the family  $\{f_x \mid x \in X\}$  is bounded by  $g$ .

"If." Assume that  $X$  has cardinality smaller than  $\mathfrak{b}$ . We show that  $\mathcal{L}(X)$  has the Egoroff property by verifying the criterion (\*) in the lemma. So let  $A_{nk}$ 's and  $A$  be given, satisfying the hypotheses of (\*). For each  $x \in A$  and each  $n$ , let  $f_x(n)$  be a  $k$  such that  $x \in A_{nk}$ . Since  $A_{nk}$  increases with  $k$ , we have  $x \in A_{nk}$  for all  $k \geq f(n)$ . The family of functions  $\{f_x \mid x \in A\}$  has cardinality no larger than that of  $X$ , hence smaller than  $\mathfrak{b}$ . So let  $g: \omega \rightarrow \omega$  bound this family. This means that, for each  $x \in X$ , we have  $f_x(n) \leq g(n)$  for all sufficiently large  $n$ . (The "sufficiently large" can depend on  $x$ .) Therefore,  $x \in A_{n, g(n)}$  for all sufficiently large  $n$ . As  $x$  is an arbitrary element of  $A$ , the sets

$$B_m = \bigcap_{n \geq m} A_{n, g(n)}$$

cover  $A$ . The sequence  $\{B_m\}$  is obviously increasing, and  $B_n \subseteq A_{n, g(n)}$ . Thus, the criterion (\*) is verified.  $\square$

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