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A NEW PROOF OF A WEIGHTED INEQUALITY FOR THE ERGODIC MAXIMAL FUNCTION

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ABSTRACT. E. Atencia and A. de la Torre proved that the ergodic maximal function operator is bounded on $L^p(\omega)$ if ω satisfies an appropriate analogue of Muckenhoupt's A_p condition. An alternate proof of this result is given.

Let (Ω, Σ, μ) be a probability space and suppose $T\colon \Omega \to \Omega$ is an ergodic, invertible measure preserving transformation. If $\sigma \in L^1(d\mu)$ with $\sigma > 0$ a.e., the two-sided ergodic maximal operator with respect to σ, denoted M_σ, is defined for nonnegative integrable f by $(M_\sigma f)(x) = \sup_{m, n \geqslant 0} A_\sigma(f; m, n, x)$ where

$$A_\sigma(f; m, n, x) = \frac{\sum_{k=-m}^{n} f(T^k x)\sigma(T^k x)}{\sum_{k=-m}^{n} \sigma(T^k x)}, \qquad x \in \Omega.$$

The one-sided maximal operators M_σ^+ and M_σ^- are defined similarly except that the supremum is taken over $m = 0$, $n \geqslant 0$ for M_σ^+ and $m \geqslant 0$, $n = 0$ for M_σ^-. If σ is the constant function equal to one, these operators will be denoted simply as M, M^+ and M^-, respectively. Note that

$$[(M_\sigma^+ + M_\sigma^-)f]/2 \leqslant M_\sigma f \leqslant (M_\sigma^+ + M_\sigma^-)f. \tag{1}$$

E. Atencia and A. de la Torre [1] proved that M is bounded on $L^p(\omega d\mu)$, $1 < p < \infty$, if $\omega \in L^1(d\mu)$ with $\omega > 0$ a.e. and

$$\left[\frac{1}{i}\sum_{k=0}^{i-1} \omega(T^k x)\right]\left[\frac{1}{i}\sum_{k=0}^{i-1} \omega^{-1/(p-1)}(T^k x)\right]^{p-1} \leqslant C \quad \text{a.e.} \tag{A_p}$$

for some constant C and all positive integers i. In this note an alternate proof of this result is given. Our proof, an adaptation of that given by M. Christ and R. Fefferman [2] for the Hardy-Littlewood maximal function operator, uses only elementary consequences of the (A_p) condition; in particular, use of the "reverse Hölder" inequality property is avoided.

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The Maximal Ergodic Theorem asserts that M^+ is of weak type $(1,1)$ with respect to $d\mu$. The elegant proof of this result given recently by R. Jones [3] is easily generalized to show that M_σ^+ is of weak type $(1,1)$ with respect to the measure $\sigma\, d\mu$; indeed,

$$\int_{\{x:\,(M_\sigma^+ f)(x)>\lambda\}} \sigma(x)\, d\mu(x) \leqslant \frac{1}{\lambda}\int_\Omega f(x)\sigma(x)\, d\mu(x). \tag{2}$$

Since T^{-1} is also ergodic and measure preserving, it follows that M_σ^- is also of weak type $(1,1)$ with respect to $\sigma d\mu$. Since these operators are clearly of strong type (∞,∞), the Marcinkiewicz interpolation theorem shows that these operators are bounded on $L^p(\sigma\, d\mu)$, $1<p<\infty$; from (1) it follows that the same is true of M_σ.

Set $\sigma(x)=\omega(x)^{-1/(p-1)}$. Observe first that if ω satisfies (A_p) then $\sigma\in L^1(d\mu)$. To see this, let $\sigma_n(x)=\min[\sigma(x),n]$ so that $\sigma_n\in L^1(d\mu)$ and (A_p) shows that

$$\left[\frac{1}{i}\sum_{k=0}^{i-1}\omega(T^k x)\right]\left[\frac{1}{i}\sum_{k=0}^{i-1}\sigma_n(T^k x)\right]^{p-1}\leqslant C \quad \text{a.e.}$$

The Dominated Ergodic and Monotone Convergence Theorems show, upon letting $i\to\infty$, then $n\to\infty$, that

$$\left[\int_\Omega \omega\, d\mu\right]\left[\int_\Omega \sigma\, d\mu\right]^{p-1}\leqslant C. \tag{3}$$

The main step in the proof is the estimate

$$\int_\Omega (M^+ f)^p \omega\, d\mu \leqslant B\int_\Omega f^p\omega\, d\mu \tag{4}$$

for if this is proved, then upon replacing T by T^{-1} we obtain a similar estimate for M^-, and (1) then yields the required estimate for M.

Without loss of generality assume that $\int_\Omega f\, d\mu = 1$ and then set $E^k=\{x\in\Omega:\ (M^+f)(x)>4^k\}$ for $k=1,2,\ldots$. Now (2) shows that $\mu(E^k)<1$ and since T is ergodic it follows that for almost all $x\in E^k$ there are positive integers $r=r(x)$ and $s=s(x)$ such that $T^jx\in E^k$ if $-r+1\leqslant j\leqslant s-1$ but $T^jx\notin E^k$ for $j=-r$ and $j=s$. Thus, if $B_i^k=\{x\in E^k:\ r(x)=1 \text{ and } s(x)=i\}$, then the sets $T^j(B_i^k)$, $0\leqslant j\leqslant i-1$, $i=1,2,\ldots$, are pairwise disjoint and their union (up to a set of measure zero) is E^k.

I wish to thank the referee for pointing out that this decomposition of E^k is the same as that which results from the Kakutani decomposition of $\Omega\setminus E^k$ and in that context the inequalities (5) below were obtained by R. Jones [4].

We need the following lemma but postpone its proof until the end of this paper:

LEMMA. *If χ_k denotes the characteristic function of $\Omega\setminus E^k$, then for $x\in B_i^k$*

$$4^k<\frac{1}{i}\sum_{j=0}^{i-1} f(T^jx)\leqslant 2\cdot 4^k \tag{5}$$

and

$$\sum_{j=0}^{i-1} \sigma(T^j x) \leqslant (2^p C)^{1/(p-1)} \sum_{j=0}^{i-1} (\sigma \chi_{k+1})(T^j x). \tag{6}$$

Now to obtain (4), write

$$\int_\Omega (M^+ f)^p \omega \, d\mu \leqslant 4^p \int_\Omega \omega \, d\mu + \sum_{k=1}^{\infty} 4^{(k+1)p} \int_{E^k \setminus E^{k+1}} \omega \, d\mu \tag{7}$$

and observe that the first term on the right has the required bound, in view of (3), since by Hölder's inequality

$$\int_\Omega \omega \, d\mu = \left[\int_\Omega \omega \, d\mu\right]\left[\int_\Omega f \, d\mu\right]^p \leqslant \left[\int_\Omega \omega \, d\mu\right]\left[\int_\Omega \sigma \, d\mu\right]^{p-1}\left[\int_\Omega f^p \omega \, d\mu\right].$$

On the other hand,

$$4^{kp} \int_{E^k \setminus E^{k+1}} \omega \, d\mu \leqslant 4^{kp} \int_{E^k} \omega \, d\mu = \sum_{i=1}^{\infty} \int_{B_i^k} 4^{kp} \sum_{j=0}^{i-1} \omega(T^j x) \, d\mu$$

and (5) shows that this is bounded by

$$\sum_{i=1}^{\infty} \int_{B_i^k} \left[\frac{1}{i} \sum_{j=0}^{i-1} f(T^j x)\right]^p \left[\sum_{j=0}^{i-1} \omega(T^j x)\right] d\mu$$

$$= \sum_{i=1}^{\infty} \int_{B_i^k} \left[A_\sigma(f\sigma^{-1}; 0, i-1)\right]^p \left[\sum_{j=0}^{i-1} \omega(T^j x)\right]\left[\frac{1}{i} \sum_{j=0}^{i-1} \sigma(T^j x)\right]^p d\mu.$$

Now (A_p) and (6) show that this is bounded by

$$(2C)^{p'} \sum_{i=1}^{\infty} \int_{B_i^k} \left[A_\sigma(f\sigma^{-1}; 0, i-1)\right]^p \left[\sum_{j=0}^{i-1} (\sigma\chi_{k+1})(T^j x)\right] d\mu$$

$$\leqslant (2C)^{p'} \sum_{i=1}^{\infty} \int_{B_i^k} \sum_{j=0}^{i-1} \left[(M_\sigma f\sigma^{-1})\right]^p (\sigma\chi_{k+1})(T^j x) \, d\mu$$

$$= (2C)^{p'} \int_{E^k \setminus E^{k+1}} \left[M_\sigma f\sigma^{-1}\right]^p \sigma \, d\mu.$$

Summing over k and using the boundedness of M_σ on $L^p(\sigma d\mu)$ shows that the second term in (7) also has the required bound. Thus we have (4).

It remains only to prove the lemma.

The right-hand inequality of (5) is clear since $T^{-1}x \notin E^k$ implies

$$\frac{1}{i+1} \sum_{j=-1}^{i-1} f(T^j x) \leqslant 4^k.$$

Since $x \in E^k$ there is a positive integer n such that

$$\sum_{j=0}^{n-1} f(T^j x) > 4^k n. \tag{8}$$

We may assume that $n \geqslant i$, for otherwise, to derive a contradiction, let n be the largest integer satisfying (8). Then, since $n < i$, $T^n x \in E^k$ so there is a positive integer m such that $\sum_{j=n}^{n+m-1} f(T^j x) > 4^k m$. Adding this to (8) then contradicts the maximality of n. On the other hand, if $n > i$ in (8), then $T^i x \notin E^k$ implies

$$\sum_{j=i}^{n-1} f(T^j x) \leqslant 4^k (n-i).$$

Subtracting this from (8) shows that (8) also holds for $n = i$. This proves (5).

Next we show that

$$\sum_{j=0}^{i-1} \chi_{k+1}(T^j x) > i/2, \qquad x \in B_i^k. \tag{9}$$

Write $J = \{j\colon 0 \leqslant j \leqslant i-1, T^j x \in E^{k+1}\}$ as a disjoint union of maximal blocks of consecutive integers J_q, $q = 1, \ldots, m$. If $|J|$ denotes the cardinal of J, then by (5) we have

$$|J| = \sum_{q=1}^{m} |J_q| < \sum_{q=1}^{m} \left[4^{-k-1} \sum_{n \in J_q} f(T^n x) \right] \leqslant 4^{-k-1} \sum_{j=0}^{i-1} f(T^j x)$$

and again by (5), this does not exceed $i/2$. This proves (9).

The (A_p) condition and (9) show that

$$\left[\sum_{j=0}^{i-1} \omega(T^j x) \right] \left[\sum_{j=0}^{i-1} \sigma(T^j x) \right]^{p-1} \leqslant C i^p \leqslant 2^p C \left[\sum_{j=0}^{i-1} \chi_{k+1}(T^j x) \right]^p$$

and then Hölder's inequality shows that this does not exceed

$$2^p C \left[\sum_{j=0}^{i-1} (\omega \chi_{k+1})(T^j x) \right] \left[\sum_{j=0}^{i-1} (\sigma \chi_{k+1})(T^j x) \right]^{p-1}.$$

From this (6) follows and the lemma is proved.

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