PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 4, December 1986

A STRENGTHENING OF LETH'S UNIQUENESS CONDITION FOR SEQUENCES

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ABSTRACT. A series $\sum a_i$ of nonnegative reals summing to 1 such that $a_i \leq \sum_{j>i} a_j$ for each i is uniquely characterized by the equalities of the form $\sum_J a_i = \sum_K a_k$. This characterization is an improvement of one given by Leth.

The main purpose of this note is to prove the following sharpened version of a theorem of S. Leth [**2**].

THEOREM. *Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be sequences of real numbers such that*
(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$,
(ii) $0 < a_{n+1} \leq a_n$ *and* $0 < b_{n+1} \leq b_n$ *for all* n,
(iii) $a_n \leq \sum_{j>n} a_j$ *and* $b_n \leq \sum_{j>n} b_j$ *for all* n,
(iv) $\sum_{j\in J} a_j = \sum_{k\in K} a_k$ *iff* $\sum_{j\in J} b_j = \sum_{k\in K} b_k$ *for all* J *and* K.
Then there is a constant u *such that* $a_i = ub_i$ *for all* i.

In Leth's theorem (iv) is replaced by
(iv)′ $\sum_{j\in J} a_j \leq \sum_{k\in K} a_k$ iff $\sum_{j\in J} b_j \leq \sum_{k\in K} b_k$ for all J and K.
J. Mycielski [**3**] asked if (iv) suffices. To see that the answer to his qustion is yes, we need several lemmas. The lemmas and their proofs are variants of those in [**1** and **2**]. N is the set of nonnegative integers.

LEMMA 1. *Let* $r \leq \sum_{i=1}^{\infty} a_i$ *where* $a_{n+1} \leq a_n \leq \sum_{j>n} a_j$ *for all* n *and* $\lim_{n\to\infty} a_n = 0$. *Then there is some* $K \subseteq N$ *such that* $r = \sum_K a_k$.

PROOF. We define $K = \{k_0, k_1, \dots\}$ inductively. Let k_0 be the least j such that $a_j \leq r$. If k_i is known for $i < n$ and $\sum_{i<n} a_{k_i} < r$ let k_n be the least j such that $a_j + \sum_{i<n} a_{k_i} \leq r$ (such a j exists since $\lim_{n\to\infty} a_n = 0$); otherwise take $K = \{k_0, \dots, k_{n-1}\}$. Clearly $\sum_{k\in K} a_k \leq r$. To see that we cannot have $\sum_{k\in K} a_k < r$ first note that the definition of K and the assumption $\lim_{n\to\infty} a_n = 0$ imply K is infinite. By assumption $\sum_{k\in N} a_k \geq r$, hence $K \neq N$ and so there is a greatest $l \notin K$. But $a_l \leq \sum_{k>l} a_k$ forcing $l \in K$—a contradiction. Therefore $\sum_{k\in K} a_k = r$.

LEMMA 2. *Under the assumptions of Lemma* 1, *for every* j *there is a* $K \subseteq \{i : i > j\}$ *such that* $a_j = \sum_K a_i$. *Hence* $a_j = \sum_L a_l$ *for some infinite* L.

PROOF. This follows from Lemma 1 by considering the sequence $\langle a_{j+1+i} \rangle_{i=0}^{\infty}$ and taking r to be a_j. Iterating this procedure on the last term of the expansion as long as the expansion is finite gives the desired infinite expansion.

Received by the editors October 15, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 60A05; Secondary 60B99.

LEMMA 3. *Under the assumptions of Lemma* 1, *for every* $J \neq N$ *there is a* K *such that* $\sum_J a_j = \sum_K a_k$ *and* $N - K$ *is infinite.*

PROOF. If $N - J$ is finite with largest member j, let $a_j = \sum_P a_i$ where P is infinite as in Lemma 2. Now let K be the complement of $((N - J) - \{j\}) \cup P$.

PROOF OF THE THEOREM. We show that (i)–(iv) imply (iv)$'$. If not, then there is a J and an L such that $\sum_L a_l < \sum_J a_j$ and $\sum_L b_l > \sum_J b_j$. Fix J and let $r = \sup\{\sum_L b_l : \sum_L a_l < \sum_J a_j$ and $\sum_L b_l > \sum_J b_j\}$. By Lemma 1 there is an M such that $\sum_M b_m = r$.

We claim that $\sum_M a_m < \sum_J a_j$. For let M' be a finite initial subset of M. Take L such that $\sum_L a_l < \sum_J a_j$, $\sum_L b_l > \sum_J b_j$, and $\sum_L b_l > \sum_{M'} b_m$. By Lemma 1 (with $r = \sum_L b_l - \sum_{M'} b_m$ and $\sum_{i=1}^{\infty} a_k$ replaced by $\sum_{i>k} a_i$ where $k = \max M'$) there is some M'' such that $M' \subseteq M''$ and $\sum_{M''} b_m = \sum_L b_l$. Hence $\sum_{M'} a_m \leq \sum_{M''} a_m = \sum_L a_l < \sum_J a_j$. Therefore $\sum_M a_l \leq \sum_J a_j$. Equalilty implies $\sum_M b_j = \sum_J b_j$ by (iv), so we must have $\sum_M a_m < \sum_J a_j$, as claimed.

By Lemma 3, we may assume that $N - M$ is infinite and since $\lim_{n\to\infty} a_n = 0$, there is a $j \in N - M$ such that $a_j + \sum_M a_m < \sum_J a_j$. But then $b_j + \sum_M b_m > r$ and also $b_j + \sum_M b_m > \sum_J b_j$, a contradiction which finishes the proof.

REFERENCES

1. R. Chauqui and J. Malitz, *Preorderings compatible with probability measures*, Trans. Amer. Math. Soc. **279** (1983), 811–824.
2. S. Leth, *A uniqueness condition for sequences*, Proc. Amer. Math. Soc. **93** (1985), 287–290.
3. J. Mycielski, Personal communication.

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