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ON INFINITE DIMENSIONAL FEATURES OF PROPER AND CLOSED MAPPINGS

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ABSTRACT. We consider some global properties of continuous proper and closed maps acting in infinite-dimensional Fréchet manifolds. These essentially infinite-dimensional features are related to the following questions: 1. When is a closed map proper? 2. When can the "singularity set" of the map, i.e. the subset of the domain of definition where the map is not a local homeomorphism, be deleted? We establish the final answer to the first question and an answer to the second one when the singular set is a countable union of compact sets.

In global nonlinear analysis one of the main questions is that of invertibility in the large of nonlinear maps which in turn is closely connected with problems of surjectivity and deletion of "singularity sets" of maps. These questions are considered in this paper for two types of maps: proper and closed ones.

In §1 we give the final unimprovable result concerning the identity of the notions "proper" and "closed" for maps acting in infinite-dimensional Fréchet manifolds.

In §2 we give conditions for surjectivity, deletion of the "singular set" of a map and find when a proper (closed) map, whose "singular set" can be a countable union of compact sets, is a global homeomorphism.

1. When is a closed map also proper? As is known (see [**1**]) under general assumptions on spaces of definition (embracing all metrizable ones) any continuous proper map $f: X \to Y$ is also closed. (Recall that f is closed if the image of any closed set is closed; f is proper if the preimage of any compact set is compact.)

Even simple examples of constant maps of the type $f(x) = 0$, where $x \in (-\infty, \infty)$, show that not each closed map is proper. Nevertheless it turns out that for infinite-dimensional manifolds the situation is considerably simpler and Smale [**2**] first announced the following infinite-dimensional condition under which the notions of properness and closedness coincide.

LEMMA (SMALE [**2**, 1965]). *Let X, Y be connected Banach manifolds, $f: X \to Y$ a Fredholm closed map. Then f is proper if* $\dim X = \infty$.

A far from trivial complete proof of this lemma was given in [**3**] where all assumptions were essential.

To formulate our result we introduce the following definition.

DEFINITION 1.1. (1) A locally convex topological vector space is a Fréchet space if it is a complete metrizable space. A Hausdorff paracompact space M locally homeomorphic to a Fréchet space X (i.e. each point of M possesses an open

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neighborhood homeomorphic to an open subset of X) is a Fréchet manifold. If $\dim X = \infty$, then M is called an infinite-dimensional Fréchet manifold. If Fréchet spaces are replaced in the above definition by Banach spaces, we get the definition of a Banach manifold.

(2) [**4**] A subset Q of a topological space X is negligible if X is homeomorphic to $X \backslash Q$.

(3) A map is called nonconstant if its image contains more than one point; the subset A of a topological space X with the topology induced by X is a subspace.

REMARK 1.1. It follows directly from the definition and [**5**] that a Fréchet manifold M is metrizable with complete metric. Hence if $M \supset A$ is a first category set, then $M \backslash A$ is dense in M. This in particular implies that if $M \supset W$ is an open nonempty subset, then $W \backslash A \neq \varnothing$.

THEOREM 1.1. *Let X, Y be Hausdorff spaces, where X is a connected infinite-dimensional Fréchet manifold, and Y satisfies the first countability axiom, and let $f: X \to Y$ be a continuous closed nonconstant map. Then f is proper.*

PROOF. Let $y \in Y$ be an arbitrary point and $f^{-1}(y) \neq \varnothing$. By continuity of f we have the following disjoint presentation

$$f^{-1}(y) = \operatorname{Fr} f^{-1}(y) \cup \operatorname{int} f^{-1}(y).$$

Suppose that $\operatorname{int} f^{-1}(y) \neq \varnothing$. Since f is nonconstant, then $X \backslash f^{-1}(y) \equiv G \neq \varnothing$ and clearly G is an open set and $G \cap \operatorname{int} f^{-1}(y) = \varnothing$. Hence $G \cup \operatorname{int} f^{-1}(y)$ is a nonconnected subset of X. Since X satisfies the first countability axiom (see Remark 1.1) then by [**6**] we see that $\operatorname{Fr} f^{-1}(y)$ is compact. Then by [**7**] $\operatorname{Fr} f^{-1}(y)$ is a negligible set, i.e. X is homeomorphic to

$$X \backslash \operatorname{Fr} f^{-1}(y) = \operatorname{int} f^{-1}(y) \cup G.$$

Taking into account the fact that X is connected we get a contradiction. Hence $\operatorname{int} f^{-1}(y) = \varnothing$ so that $f^{-1}(y) = \operatorname{Fr} f^{-1}(y)$ is compact. Thus, f is a continuous closed map and $f^{-1}(y)$ is compact for any $y \in Y$. Then by [**1**] f is a proper map.

The following simple example of a closed nonconstant map $f: (-\infty, +\infty) \to (-\infty, +\infty)$ defined by

$$f(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x \leq 1, \\ 1, & 1 < x, \end{cases}$$

shows that Theorem 1.1 fails in finite-dimensional spaces.

It is not difficult to see that in Theorem 1.1 spaces X and Y can be, in particular, Banach manifolds. Since for infinite-dimensional Banach manifolds a Fredholm map cannot be constant, the theorem immediately implies the Smale lemma.

2. On infinite-dimensional conditions for surjectivity, "deletion" of singularity sets and global homeomorphicity of maps. Let X, Y be topological spaces, $f: X \to Y$ a map. Denote by B_f the set of points of X at which f is not a local homeomorphism and by P_f a closed subset of X such that at each point of $X \backslash P_f$ the map f is a local homeomorphism. Points of B_f are called singular points or just singularities and the set B_f itself the singular set; points of P_f are called pseudosingular ones or just pseudosingularities of f and P_f itself is called a pseudosingular set.

The introduction of nonuniquely defined sets P_f is of technical character and the naive explanation is that, generally speaking, sets B_f are "more difficult" to describe for a given map than sets of type P_f. For instance let $x = (x_1, x_2, x_3)$, $f: \mathbf{R}^3 \to \mathbf{R}^3$, $f(x) = (f_1(x), f_2(x), f_3(x))$:

$$f_1(x) = \tfrac{1}{3}x_1^3 + x_1x_2^2 + x_1x_3^2, \quad f_2(x) = x_2, \quad f_3(x) = x_3.$$

Since $\det f'(x) = x_1^2 + x_2^2 + x_3^2$, then f is a local homeomorphism in $\mathbf{R}^3 \backslash \{0\}$ for $x \neq 0$. Hence, as P_f we can take e.g. $\{0\}$. Actually, in this example $B_f = \varnothing$, which is already not too obvious.

It is obvious that B_f is always a set of type P_f and if an arbitrary set P_f is fixed, then $B_f \subset P_f$. The definitions also imply that B_f, P_f are closed subsets of X.

Let ψ be a family of closed subsets of X. Let us say that $f: X \to Y$ possesses the deletion of the singular set property with respect to the family ψ if $B_f = \varnothing$ whenever $P_f \in \psi$.

The following question naturally arises: Under what conditions does f possess the deletion of the singular set property with respect to the given family ψ?

Fixing ψ we get a concrete problem which requires a special approach to its solution. For instance, in the case when X, Y are normed spaces of dimension no less than 3 and $f: X \to Y$ is a locally proper (closed) continuous map in [**8**] (only in the finite dimensional case); and in [**9**] the following answer is given: f possesses the deletion of the singular set property with respect to ψ consisting of closed discrete subsets.

(Recall that $f: X \to Y$ is locally proper (closed) if each point $x \in X$ possesses a closed neighborhood U such that the restriction of f onto U is a proper (closed) map from a subspace U to Y. Clearly each proper (closed) map is a locally proper (closed) one since as U we can take the whole X.)

In [**8**] a result is also obtained concerning deletion of the singular set with respect to the family ψ consisting of compacts, namely the following

THEOREM (CHURCH, HEMMINGSEN [**8**, 1960]). *Let f be an open map of n-dimensional Euclidean space E^n onto E^n, $n \neq 2$, and assume that* $\dim f(B_f) \leq n-2$. *If the restriction of f to $E^n \backslash f^{-1}(f(B_f))$ is a covering map and B_f is compact, then f is a homeomorphism.*

The following theorem is a far-reaching extension of this result. It answers the above question when ψ consists of closed subsets of X, each of them being a countable union of compacts.

THEOREM 2.1. *Let X, Y be connected infinite-dimensional Fréchet manifolds, Y simply connected, $f: X \to Y$ a continuous map, P_f a finite or countable union of compact sets, and one of the following two conditions holds*:

(a) *f is a closed map*;

(b) *f is a proper map.*

Then (1) *$f(X) = Y$, $f^{-1}(f(P_f)) = P_f$, and the restriction of f to $X \backslash P_f$ is a homeomorphism of $X \backslash P_f$ onto $Y \backslash f(P_f)$*;

(2) *if f is an open map, then $B_f = \varnothing$ and f is a global homeomorphism of X onto Y.*

PROOF. First notice that if A is a compact subset of X, then A is nowhere dense in X. This follows immediately from the definition of infinite-dimensional Fréchet manifolds and the fact that any locally compact topological vector space (real or complex) is finite dimensional. Hence, $X\backslash P_f \neq \varnothing$ by Remark 1.1. It follows that f is a nonconstant map and so, thanks to Theorem 1.1 and [**1**], f is both proper and closed.

Notice also that any Fréchet manifold is clearly locally linearly connected. Therefore X, Y, being connected, are also linearly connected spaces.

Now let us pass to the proof of the first part of the theorem.

By hypothesis we have $P_f = \bigcup_{i=1}^{\infty} A_i$, where A_i is a compact subset of X. Due to the continuity of f we have

$$f(P_f) = \bigcup_{i=1}^{\infty} f(A_i)$$

is a countable union of compacts in Y and therefore by [**7**] $f(P_f)$ is a negligible subset of Y. Then we see that $Y\backslash f(P_f)$ is connected and simply connected.

Since P_f is a closed set and f is a closed map, then $f(P_f)$ is a closed subset of Y. Thus, by the local linear connectedness of Y, we see that $Y\backslash f(P_f)$ is a simply connected linearly connected domain.

Since f is continuous and proper,

$$f^{-1}(f(P_f)) = \bigcup_{i=1}^{\infty} f^{-1}(f(A_i))$$

is a closed set which is the union of a countable set of compacts. Then by [**7**], $f^{-1}(f(P_f))$ is a negligible subset of X, and therefore $X\backslash f^{-1}(f(P_f))$ is a linearly connected domain.

Since $f^{-1}(Y\backslash f(P_f)) = X\backslash f^{-1}(f(P_f))$, f is a proper map from $X\backslash f^{-1}(f(P_f))$ to $Y\backslash f(P_f)$.

Since the restriction of f to $X\backslash f^{-1}(f(P_f))$ is a local homeomorphism, this restriction is a homeomorphism of $X\backslash f^{-1}(f(P_f))$ onto $Y\backslash f(P_f)$ (see [**10**]).

Let us show that $f^{-1}(f(P_f)) = P_f$. Let $y \in f(P_f)$; hence for some $a \in P_f$ we have $y = f(a)$.

Suppose that there exists a point $c \in X\backslash P_f$ such that $y = f(c)$. Then since X is Huasdorff, P_f is closed, and f is a local homeomorphism at c, the points a and c possess disjoint open neighborhoods U_a and $U_c \subset X\backslash P_f$ respectively and the restriction of f to U_c is a homeomorphism of U_c onto an open neighborhood $f(U_c)$ of y.

By the local linear connectedness of X we may assume that U_a and U_c are connected neighborhoods. We have

$$U_a\backslash \bigcup_{i=1}^{\infty} f^{-1}(f(A_i)) = U_a\backslash \bigcup_{i=1}^{\infty} (U_a \cap f^{-1}(f(A_i))).$$

Since f is proper and $f(A_i)$ is compact, $f^{-1}(f(A_i))$ is compact for $i \in \mathbf{N}$.

Set $T_i = (f^{-1}(f(A_i)) \cap U_a)\backslash\{a\}$, so that $\{T_i, i \in \mathbf{N}\}$ is a no more than a countable family of locally compact subsets of U_a. We have

$$U_a\backslash \bigcup_{i=1}^{\infty} T_i = W_a, \qquad W_a \ni a,$$

and by [**7**] $\bigcup_{i=1}^{\infty} T_i$ is a negligible subset of U_a, i.e. U_a is homeomorphic to W_a. Denote by g a corresponding homeomorphism from U_a onto W_a and let $g^{-1}(a) = \tilde{a} \in U_a$.

It is easy to see that a connected open subspace U_a has no isolated points, hence $\tilde{a}$ is not an isolated point of U_a satisfying the first countability axiom. Thus there exists a sequence $\tilde{a}_n \in U_a$ such that $\tilde{a}_n \to \tilde{a}$, $\tilde{a}_n \neq \tilde{a}$. Since g is a homeomorphism, this implies

$$g(\tilde{a}_n) = a_n \to a = g(\tilde{a}), \quad a_n \neq a, n \in \mathbf{N}.$$

We have $W_a \cap f^{-1}(f(P_f)) = \{a\}, a \in W_a$. Therefore $a_n \in X \backslash f^{-1}(f(P_f))$ and

$$f(a_n) \in f(X \backslash f^{-1}(f(P_f))) = f(X) \backslash f(P_f),$$

i.e. $f(a_n) \notin f(P_f)$.

By the continuity of f, we have $f(a_n) \to f(a) = y$, $n \in \mathbf{N}$; hence starting from some n_0 we have $f(a_n) \in f(U_c)$, $n \geq n_0$, $n \in \mathbf{N}$. Then since $f|U_c$ is a homeomorphism of U_c onto $f(U_c)$, we deduce with necessity the existence of a sequence $b_n \in U_c$ such that $f(b_n) = f(a_n) \in f(U_c)$, where clearly $b_n \neq a_n$ as $n \geq n_0$.

Since $f(a_n) \notin f(P_f)$, then $b_n \notin f^{-1}(f(P_f))$ and therefore

$$a_n, b_n \in X \backslash f^{-1}(f(P_f)).$$

This contradicts the proved injectivity of the restriction of f onto $X \backslash f^{-1}(f(P_f))$.

Thus for an arbitrary $y \in f(P_f)$ we have $f^{-1}(y) \cap (X \backslash P_f) = \varnothing$, i.e. $f^{-1}(f(P_f)) = P_f$ and therefore the restriction of f to $X \backslash P_f$ is a homeomorphism of $X \backslash P_f$ onto $Y \backslash f(P_f)$.

Combining this with what was proved above, we finally have

$$f(X) = f(P_f \cup (X \backslash P_f)) = f(P_f) \cup (Y \backslash f(P_f)) = Y$$

and the first part of the theorem is completely proved.

Let us now prove the second part of the theorem. By what we have already proved, the restriction of f to $X \backslash P_f$ is an injective map of $X \backslash P_f$ onto $Y \backslash f(P_f)$ and $f^{-1}(f(P_f)) = P_f$. Therefore if we prove that the restriction of f to P_f is also injective, then we will get that f is a bijective continuous closed map of X onto Y, i.e. f is a homeomorphism of X onto Y and $B_f = \varnothing$.

Assume the contrary, i.e. let $x_1, x_2 \in f^{-1}(y) \subset P_f$, $x_1 \neq x_2$, for some point $y \in f(P_f)$.

Since f is open and X is Hausdorff, we have disjoint open neighborhoods U_1 and U_2 of x_1 and x_2 respectively such that $V = f(U_1) \cap f(U_2)$ is an open neighborhood of y.

Since V is an open subset of Y and $f(P_f)$ is a countable union of compacts, then $V \backslash f(P_f) \neq \varnothing$ (see Remark 1.1) and therefore in $V \backslash f(P_f)$ there is a point, say z, and points $x_1' \in U_1$ and $x_2' \in U_2$ such that $f(x_1') = f(x_2') = z$. Since $z \notin f(P_f)$ and $f^{-1}(f(P_f)) = P_f$, then $f^{-1}(z) \cap P_f = \varnothing$ and therefore $x_1' \in U_1 \backslash P_f$, $x_2' \in U_2 \backslash P_f$.

Thus $x_1', x_2' \in X \backslash P_f$, $f(x_1') = f(x_2') = z$. Therefore we have obtained a contradiction, since as we have already proved the restriction of f to $X \backslash P_f$ is injective.

REMARK 2.1. (a) Theorem 2.1 contains an essential sharpening of the Banach-Mazure criterion for maps to be globally homeomorphic [**10**] in the case of infinite-dimensional Fréchet manifolds, since it allows us to avoid the assumption that maps are locally homeomorphic on countable unions of compacts from the space of definition of these maps.

(b) In interesting papers by Berger and Plastok [**11**, **12**] problems of deletion of discrete singularities of maps in Banach spaces were investigated starting from dimension 3, but under the special additional assumption that onto maps are Fredholm. This assumption is removed in our paper. Also, for Fredholm maps Plastok [**12**] was the first to obtain the first part of Theorem 2.1.

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REFERENCES

1. M. Henriksen and J. Isbell, *Some properties of compactifications*, Duke Math. J. **25** (1958), 83–106.
2. S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.
3. Yu. G. Borisovich, W. G. Zvyaguin and Yu. I. Sapronov, *Non-linear Fredholm mappings and Leray-Shauder theory*, Russian Math. Surveys **32** (1977), no. 4, 3–54. (Russian)
4. J. Dugundji and A. Granas, *Fixed point theory*, PWN, Warsaw, 1982.
5. E. Michael, *Local properties of topological spaces*, Duke Math. J. **21** (1954), 163–171.
6. ——, *A note on closed maps and compact sets*, Israel J. Math. Sect. F **2** (1964), 173–176.
7. W. Cutler, *Negligible subsets of infinite dimensional Fréchet manifolds*, Proc. Amer. Math. Soc. **23** (1969), 668–675.
8. P. Church and E. Hemmingsen, *Light open maps on n-manifolds*, Duke Math. J. **27** (1960), 527–536.
9. R. S. Sadyrkhanov, *Condition for local homeomorphy of a mapping*, Math. USSR Dokl. **275** (1984), no. 6, 1316–1320. (Russian)
10. S. Banach and S. Mazur, *Über mehrdeutige stetige Abbildungen*, Studia Math. **5** (1934), 174–178.
11. M. Berger and R. Plastock, *On the singularities of non-linear Fredholm operators*, Bull. Amer. Math. Soc. **83** (1977), 1316–1318.
12. R. Plastock, *Nonlinear Fredholm maps of index zero and their singularities*, Proc. Amer. Math. Soc. **68** (1978), 317–322.

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