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ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF THE SECOND ORDER DIFFERENCE EQUATION

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ABSTRACT. The second order difference equation

$$\text{(E)} \qquad \Delta^2 x_n + p_n f(x_n) = 0$$

is considered. The results give a necessary and sufficient condition for some solution of (E) to have asymptotic behavior $x_n \sim C =$ const. as n approaches infinity.

Introduction. The asymptotic behavior of the solutions of second order differential equations have been considered by R. A. Moore and Z. Nehari [**4**], W. F. Trench [**9**], and P. Waltman [**10**]. The next results for nth order nonhomogeneous differential equations was given by T. G. Hallam [**1**, **2**]. Similar problems with regard to second order difference equations were investigated by J. W. Hooker and W. T. Patula [**3**] and J. Popenda [**7**].

In this paper the asymptotic behavior of solutions of the second order difference equation

$$\text{(E)} \qquad \Delta^2 x_n + p_n f(x_n) = 0$$

will be considered. A necessary and sufficient condition for some solution x of (E) to have the asymptotic behavior

$$\text{(AB)} \qquad \lim_{n\to\infty} x_n = C,$$

where C is a constant such that $f(C) \neq 0$, will be proved.

Let N denote the set of positive integers and R the set of real numbers. Throughout this paper it will be assumed that $f\colon R \to R$ is continuous and $p\colon N \to R_+ \cup \{0\}$.

For a function $a\colon N \to R$ we introduce the difference operator Δ by

$$\Delta a_n = a_{n+1} - a_n, \quad \Delta^2 a_n = \Delta(\Delta a_n),$$

where $a_n = a(n)$, $n \in N$. Moreover let $\sum_{j=k}^{k-1} a_j = 0$. One can observe that if f is definite and finite on R then (E) possesses solutions for any two initial values $x_1, x_2 \in R$.

1. A necessary condition.

THEOREM 1. *A necessary condition for the existence of a solution x of* (E) *which possesses asymptotic behavior* (AB) *is*

$$\text{(NS)} \qquad \sum_{j=1}^{\infty} j p_j < \infty.$$

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PROOF. Let x denote a solution of (E) having the property (AB), i.e. $x_n \to C$ for $n \to \infty$. Then

$$\Delta x_n \to 0 \quad \text{as } n \to \infty. \tag{1.1}$$

Assume that $f(C) > 0$. (The case $f(C) < 0$ with some modifications can be considered in a similar way.) The continuity of f implies that there exists $\varepsilon > 0$ such that $f(t) > 0$, $t \in I := [C-\varepsilon, C+\varepsilon]$ for some $\varepsilon > 0$. Since $x_n \to C$ as $n \to \infty$, there exists $n_1 = N(\varepsilon)$ such that for each $n \geq n_1$, $x_n \in [C-\varepsilon, C+\varepsilon]$. Therefore

$$f(x_n) \geq C_0 := \min_{t\in I} f(t) > 0 \quad \text{for } n \geq n_1.$$

Hence

$$\Delta x_n - \Delta x_k = -\sum_{j=k}^{n-1} p_j f(x_j) \leq -C_0 \sum_{j=k}^{n-1} p_j \quad \text{for } k \geq n_1.$$

Using (1.1) we get

$$C_0 \sum_{j=k}^{\infty} p_j \leq \Delta x_k \quad \text{for } k \geq n_1. \tag{1.2}$$

Therefore the series $\sum_{j=k}^{\infty} p_j$ is convergent. Summing (1.2) over n and tending to infinity with an upper limit we yield

$$C_0 \sum_{j=n_1}^{\infty} \sum_{i=j}^{\infty} p_i \leq C - x_{n_1}. \tag{1.3}$$

From this fact it follows that the series $\sum_{j=n_1}^{\infty} \sum_{i=j}^{\infty} p_i$ converges. Since

$$\sum_{j=n_1}^{\infty} \sum_{i=j}^{\infty} p_i = \sum_{j=n_1}^{\infty} (j+1-n_1)p_j,$$

the series $\sum_{j=n_1}^{\infty} (j+1-n_1)p_j$ is also convergent. By observing that

$$\sum_{j=n_1}^{\infty} jp_j = \sum_{j=n_1}^{\infty} (j+1-n_1)p_j + (n_1-1)\sum_{j=n_1}^{\infty} p_j$$

we see that the series $\sum_{j=n_1}^{\infty} jp_j$ is convergent. Q.E.D.

REMARK 1. From (1.2) it follows that $\Delta x_k \geq 0$ for $k \geq n_1$. Therefore the solution x_n is increasing for $n \geq n_1$. We see that $x_l \leq C$ for $l \geq n_1$. This result means that if $f(C) > 0$ then the solution of (E) which possesses the asymptotic behavior (AB) monotonically approaches C from below. If $f(C) < 0$, then x_n must monotonically tend to C from above.

2. A sufficient condition.

THEOREM 2. *For every $k \in N$ let*
(*) *$i_R + p_k f\colon R \twoheadrightarrow R$ be a surjection (i_R denotes an identity function on R).*
A sufficient condition for the existence of a solution x of (E) *which possesses the asymptotic behavior* (AB) *is* (NS).

PROOF. The cases $C > 0$ and $f(C) > 0$ will be considered. (The other cases, i.e. $C < 0$ or $f(C) < 0$, with some modifications can be shown in a similar way.)

Let (NS) hold. Hence

$$\lim_{n\to\infty}\sum_{j=n}^{\infty} jp_j = 0. \tag{2.1}$$

One can observe that the sequence $\{\sum_{j=n}^{\infty} jp_j\}_{n=1}^{\infty}$ is nonincreasing. Analogous to the proof of Theorem 1 there exists an interval $I = [C-\varepsilon, C+\varepsilon]$ such that $f(t) > 0$, $t \in I$ for some $\varepsilon > 0$. Denoting $C_1 := \max_{t\in I^-} f(t)$, where $I^- = [C-\varepsilon, C]$ from (2.1), we obtain

$$C_1 \sum_{j=n}^{\infty} jp_j \le \varepsilon \quad \text{for all } n \ge N(\varepsilon).$$

Let us set

$$n_2 = \min\left\{ n \in N : C_1 \sum_{j=n}^{\infty} jp_j \le \varepsilon \right\}.$$

Let l_∞ denote the Banach space of bounded sequences $x = \{h_i\}_{i=1}^{\infty}$ with norm $\|x\| = \sup_{i\ge 1} |h_i|$. Moreover let us define the set $T \subset l_\infty$ in the following way:

$$x = \{h_i\}_{i=1}^{\infty} \in T \quad \text{if} \quad \begin{cases} h_k = C & \text{for } k = 1, 2, \ldots, n_2 - 1, \\ h_k \in I_k^- & \text{for } k \ge n_2, \end{cases}$$

where

$$I_k^- := \left[C - C_1 \sum_{j=k}^{\infty} jp_j, C \right], \qquad k \ge n_2.$$

It is easy to show that T is bounded, convex and closed in l_∞. We will show that T is compact. Set $\operatorname{diam}[a, b] = b - a$; $a, b \in R$. By (NS) it follows that $\operatorname{diam} I_n^- \to 0$ for $n \to \infty$. Choose any $\varepsilon_1 > 0$. If ε_1 is such that $\operatorname{diam} I_{n_2}^- < \varepsilon_1$, then the element $v = \{C, C, C, \ldots\} \in l_\infty$ is an ε_1-net. The case $\operatorname{diam} I_{n_2}^- \ge \varepsilon_1$ will be considered. Let $n_3 \ge n_2$ be such that $\operatorname{diam} I_{n_3}^- \ge \varepsilon_1$ and $\operatorname{diam} I_{n_3+1}^- < \varepsilon_1$. (Everyone can find n_3 because $\operatorname{diam} I_n^- \to 0$ for $n \to \infty$.) Then it is easy to show that the set of elements of the space l_∞ in the form

$$v_{s_1, s_2, \ldots, s_{n_3-n_2+1}}^{1,2,\ldots,n_3-n_2+1} = \{C, \ldots, \underset{\substack{\updownarrow \\ \text{in} \\ \text{position} \\ n_2-1}}{C}, C - s_1\varepsilon_1, \ldots, C - s_{n_3-n_2+1}\varepsilon_1, C, \ldots\}$$

where

$$s_i = 0, 1, \ldots, r_i := \operatorname{En}\left[\frac{\operatorname{diam} I_{n_2+i-1}^-}{\varepsilon_1} \right] + 1, \qquad i = 1, 2, \ldots, n_3 - n_2 + 1,$$

to set up an ε_1-net. (En denotes an entire function.) One can observe that

$$\operatorname{card}\{v_{s_1, s_2, \ldots, s_{n_3-n_2+1}}^{1,2,\ldots,n_3-n_2+1}\} = \prod_{i=1}^{n_3-n_2+1} (r_i + 1) < \infty.$$

Hence the ε_1-net is finite and by the Hausdorff theorem T is compact.

Let us define the operator A on T in the following way:

$$Ax = y = \{b_1, b_2, \ldots, b_{n_2-1}, b_{n_2}, \ldots, b_k, \ldots\},$$

where

$$b_n = \begin{cases} C & \text{for } n = 1, 2, \dots, n_2 = 1; \\ C - \sum_{j=n}^{\infty}(j+1-n)p_j f(h_j) & \text{for } n \geq n_2. \end{cases}$$

We will show that A is a function from T to T. By observing that $I_k^- \subset I^-$ it follows that $0 < f(h_k) \leq C_1$ for $k \geq n_2$. For $j \geq k$ one obtains the inequality

$$0 < (j+1-k)p_j f(h_j) \leq jp_j f(h_j) \leq C_1 jp_j.$$

Hence

$$C \geq C - \sum_{j=k}^{\infty}(j+1-k)p_j f(h_j) \geq C - C_1 \sum_{j=k}^{\infty} jp_j.$$

Thus $b_k \in I_k^-$ for $k \geq n_2$. Therefore $y \in T$.

Next we will show that A is continuous. Since f is continuous on R, it is uniformly continuous on I^-. Hence for each $\varepsilon_2 > 0$ there exists $\delta_1 > 0$ such that the condition $|t_1 - t_2| < \delta_1$ implies $|f(t_1) - f(t_2)| < \varepsilon_2$. Consider the sequence $\{x^m\}_{m=1}^{\infty}$, $x^m \in T$, such that

$$\|x^m - x^0\| \to 0; \quad \text{i.e.,} \ \sup_{n\geq 1} |h_n^m - h_n^0| \to 0, \quad \text{as } m \to \infty. \tag{2.2}$$

From (2.2) it follows that there exists $n_3 = N(\delta_1)$ such that

$$\|x^m - x^0\| < \delta_1; \quad \text{i.e.,} \ \sup_{n\geq 1} |h_n^m - h_n^0| < \delta_1 \quad \text{for } m \geq n_3.$$

Hence

$$\forall_{m\geq n_3} \ \forall_{i\in N} \ |h_i^m - h_i^0| < \delta_1.$$

Then for $m \geq n_3$

$$\begin{aligned} \|Ax^m - Ax^0\| &= \sup_{n\geq 1} |b_n^m - b_n^0| \\ &= \sup_{n\geq n_2} \left| \sum_{j=n}^{\infty}(j+1-n)p_j f(h_j^m) - \sum_{j=n}^{\infty}(j+1-n)p_j f(h_j^0) \right|, \end{aligned}$$

where $b^0 = Ax^0$ and $b^m = Ax^m$.

Since the series $\sum_{j=n}^{\infty}(j+1-n)p_j f(h_j^m)$, $\sum_{j=n}^{\infty}(j+1-n)p_j f(h_j^0)$ are convergent,

$$\|Ax^m - Ax^0\| \leq \varepsilon_2 \sum_{j=n_2}^{\infty} (j+1-n_2)p_j, \qquad m \geq n_3.$$

Hence A is continuous.

By the Schauder fixed point theorem [**8**] there exists a solution in T of the equation $x = Ax$. Let $z = \{d_1, d_2, \dots, d_{n_2-1}, d_{n_2}, \dots\}$ denote such a solution. Since $z \in T$, it can be written as follows:

$$z = \{C, C, \dots, C, d_{n_2}, d_{n_2+1}, \dots\}$$

and

$$Az = \left\{ C, C, \dots, C, C - \sum_{j=n_2}^{\infty} (j+1-n_2) p_j f(d_j), \right.$$

$$\left. C - \sum_{j=n_2+1}^{\infty} (j-n_2) p_j f(d_j), \dots \right\}.$$

Therefore

$$d_n = C - \sum_{j=n}^{\infty} (j+1-n) p_j f(d_j) \quad \text{for } n \geq n_2. \tag{2.3}$$

Applying the operator Δ to (2.3) we yield

$$\Delta d_n = \sum_{j=n}^{\infty} p_j f(d_j) \quad \text{for } n \geq n_2.$$

Hence $\Delta^2 d_n = -p_n f(d_n)$ holds for $n \geq n_2$. This means that the sequence $\{d_n\}_{n=1}^{\infty}$ fulfills the equation (E) but for $n \geq n_2$ only.

We now prove the existence of the solution $\{x_n\}_{n=1}^{\infty}$ of (E) such that $x_n = d_n$ for $n \geq n_2$.

One can observe that (E) can be rewritten as

$$x_n + p_n f(x_n) = -x_{n+2} + 2x_{n+1}.$$

If $n = n_2 - 1$ we get

$$x_{n_2-1} + p_{n_2-1} f(x_{n_2-1}) = -x_{n_2+1} + 2x_{n_2}. \tag{2.4}$$

But we demand for x_n to be equal to d_n for $n \geq n_2$.

From (2.4) we obtain

$$x_{n_2-1} + p_{n_2-1} f(x_{n_2-1}) = -d_{n_2+1} + 2d_{n_2}.$$

By $(*)$ it follows that the equation

$$x + p_{n_2-1} f(x) = -d_{n_2+1} + 2d_{n_2}$$

possesses solutions. Let us denote one of them by x_{n_2-1}. Analogously we can calculate $x_{n_2-2}, x_{n_2-3}, \dots, x_2, x_1$ one after the other. Consequently we get the sequence which fulfills (2.4), i.e. which also fulfills (E). Moreover this sequence is identical to $\{d_n\}_{n=1}^{\infty}$ for $n \geq n_2$ and it has the asymptotic behavior (AB) because $\lim_{n\to\infty} d_n = C$. Q.E.D.

REMARK 2. One can observe that if f is bounded on R or fulfills the condition $xf(x) > 0$ for $x \neq 0$ then condition $(*)$ is satisfied. From the proof of Theorem 2 we can deduce that $(*)$ may be weakened as follows:

$$i_R + p_k f \colon R \twoheadrightarrow R \quad \text{for } k < n_2,\ k \in N.$$

REMARK 3. If the assumptions of Theorem 2 hold then analogously an existence of a solution of the equation

$$\Delta^2 x_n + p_{n+k} f(x_{n+k}) = 0, \qquad k \geq 1, \tag{E$_k$}$$

having the asymptotic behavior (AB) may be proved. In this case the operator A similar to the above but with

$$b_n = C - \sum_{j=n+k}^{\infty} (j+1-n-k)p_j f(h_j) \quad \text{for } x = \{h_i\}_{i=1}^{\infty} \in T$$

should be defined.

REMARK 4. If (E) possesses a solution x such that $\lim_{n\to\infty} x_n = C$ then equation (E) has a solution with $\lim_{n\to\infty} x_n = C_2$, where $C_2 \in (C-\varepsilon, C+\varepsilon) \subset I$.

REMARK 5. If for some C, $f(C) = 0$, then independently of the form of p, equation (E) has a solution with (AB). It has the form $x_n = C$ for each $n \geq 1$. Conversely, if, for each $n \geq n_2$, $x_n = C$ is the solution of (E) then $p_n f(C) = 0$ for $n \geq n_2$. Hence $f(C) = 0$ or $p_n = 0$ for each $n \geq n_2$. For the second case ($p_n = 0$) the condition $\sum_{j=1}^{\infty} jp_j < \infty$ obviously holds.

EXAMPLE. The special case $f(x) = x$ and $k = 1$ will be studied. In this case the equation ($\mathrm{E_k}$) can be written in the following two equivalent forms:

$$(\mathrm{E}_1) \qquad \Delta^2 x_n + p_{n+1}x_{n+1} = 0, \quad x_{n+2} - q_n x_{n+1} + x_n = 0,$$

where $q_n = 2 - p_{n+1}$, $n \in N$. If $q_n < 2$, $n \in N$ and $\sum_{j=2}^{\infty}(2-q_{j-1})j < \infty$ then (E_1) possesses a solution which asymptotically approaches any positive constant.

Analogously in the case $k = 2$ one obtains the equation

$$(\mathrm{E}_2) \qquad x_{n+2} - 2q_n x_{n+1} + q_n x_n = 0,$$

where $q_n = 1/(p_{n+2}+1)$.

If $0 < q_n < 1$ and $\sum_{j=3}^{\infty}(1/q_{j-1} - 1)j < \infty$ then (E_2) possesses a solution which asymptotically approaches any positive constant.

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