PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 1, January 1987

ELEMENTARILY EQUIVALENT FIELDS WITH INEQUIVALENT PERFECT CLOSURES

CARLOS R. VIDELA

ABSTRACT. We give a counterexample to the following conjecture due to L. V. den Dries: Let F, L be two fields of characteristic p. If $F \equiv L$ then $F^{1/p^\infty} \equiv L^{1/p^\infty}$.

Introduction. Van den Dries conjectures in [**3**, Stellingen 4] that elementarily equivalent fields have elementarily equivalent perfect closures. We will give a counterexample using a predicate introduced by Cherlin [**1**] in the study of definability in power series fields of nonzero characteristic.

Our counterexample is as follows: Fix a prime number p. Let $\tilde{\mathbf{F}}_p$ be the algebraic closure of the prime field $\mathbf{F}_p$, let $K_0 = \tilde{\mathbf{F}}_p(t)$ be the rational function field, and let $L_0 = K_0^*$ be a nonstandard extension of K_0, e.g. an ultrapower of K_0. In particular $K_0 \equiv L_0$. We will prove:

THEOREM. *The perfect closures of K_0 and L_0 are not elementarily equivalent.*

The idea behind the proof is the following: we will relate the p^nth powers of t, for n an integer, to the p^nth roots of t in the perfect closure $K = K_0^{p^{-\infty}}$ of K_0. However the p^Nth powers of t with N infinite will not correspond to roots of t in $L = L_0^{p^{-\infty}}$.

We will produce a sentence Φ whose meaning is approximately

"For all n if t^{p^n} exists then t^{1/p^n} exists."

Φ will hold in K and fail in L.

We make extensive use of the polynomial $\tau(x) = x^p - x$. We will need first order definitions of the following three predicates defined in K.

(1) Con(x): "$x \in \tilde{\mathbf{F}}_p$,"

(2) Reg(x): "$\neg\exists b \in \tilde{\mathbf{F}}_p - \{0\}$, $bx \in \tau[K]$,"

(3) Link(y, x): "$\exists n \in \mathbf{Z}$, $a \in \tilde{\mathbf{F}}_p$, $y = x^{p^n} + a$ & Reg(x)."

For Con(x) we can do the following: choose $n > 2$ and relatively prime to p. Then the curve E defined by the equation $x^n + y^n = 1$ is nonrational [**2**, p. 7]. This shows that the only solutions to the equation in K_0 must be constants. The same is true of K: for if $x_0, y_0 \in K$ are such that $x_0^n + y_0^n = 1$, then by taking p^kth powers for k large enough, we get

$$\left(x_0^{p^k}\right)^n + \left(y_0^{p^k}\right)^n = 1$$

Received by the editors September 16, 1985. This paper was presented at the meeting of the Mid-Atlantic Mathematical Logic Seminar, Institute for Advanced Study, Princeton, New Jersey, December 15, 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03C60, 12L12.

Key words and phrases. Elementary equivalence, valuation, nonstandard model.

with $x_0^{p^k}$, $y_0^{p^k} \in K_0$. Hence x_0 and y_0 are constants. We have shown that Con(x) defines the same set in K_0 and in K, namely $\tilde{\mathbf{F}}_p$. Reg(x), read "x is regular," is visibly first order definable. We will use the following fact: Let $x \in K_0$. Then $K_0 = \text{Reg}(x) \Leftrightarrow K \models \text{Reg}(x)$. A proof of this is implicit in Lemma 1 below.

The definition of Link(y, x) is as follows: First define a predicate $L_0(y, x)$ in the following way:

$$\forall a \in \tilde{\mathbf{F}}_p \exists! b \in \tilde{\mathbf{F}}_p (ay - bx \in \tau[K]).$$

Notice that $L_0(y, x)$ implies x is regular: take $a = 0$. When $L_0(y, x)$ holds, we write $y[a]$ for the element b satisfying $ay - bx \in \tau[K]$, and we define Link(y, x) by a first order formalization of

$$\text{"}L_0(y, x) \,\&\, \forall a_1, a_2 \in \tilde{\mathbf{F}}_p \; (y[a_1 a_2] = y[a_1] y[a_2])\text{."}$$

We will prove later that this has the intended meaning in K. Notice also that $\tau[K]$ is an additive abelian group and that, if $L_0(y, x)$, then the map $a \mapsto y[a]$ is an abelian group homomorphism (so that Link(y, x) says it is a field isomorphism too).

We can now write down the sentence Φ:

$$\begin{aligned}\forall x, y\big([&\text{Link}(y, x) \,\&\, \neg\text{Con}(y - x) \,\&\, \forall a(\text{Con}(a) \Rightarrow \text{Reg}(x(y - a)))]\\ &\Rightarrow \exists z, b\big[\text{Con}(b) \,\&\, \text{Link}(z, x) \,\&\, \text{Link}(xz, x(y - b)) \,\&\, \neg\text{Con}(y - z)\big]\big).\end{aligned}$$

We will show that Φ is true in K; on the other hand taking $x = t$ we will see that Φ is false in L. It turns out that the content of Φ is approximately

$$\forall x \forall n \neq 0 \forall y \in x^{p^n} + \tilde{\mathbf{F}}_p\Big(\exists z, a \in \tilde{\mathbf{F}}_p\Big((xz)^{p^n} = x(y - a) \,\&\, x - z^{p^n} \in \tilde{\mathbf{F}}_p\Big)\Big).$$

1. The interpretation of Φ in K.

LEMMA 1. *For $x, y \in K_0$, if $K \models \text{Link}(y, x)$, then $K_0 \models \text{Link}(y, x)$.*

PROOF. Let $a \in \tilde{\mathbf{F}}_p$. Then there exist $b \in \tilde{\mathbf{F}}_p$ and $w \in K$ such that $ay - bx = w^p - w$. Note that w is inseparable over K_0. Since $w^p - w \in K_0$, it follows that $K_0(w) = K_0(w^p) = K_0(w^{p^2}) = \cdots$. Hence w is separable over K_0 and so must be in K_0.

We will show that Link(y, x) has the intended meaning by proving that the map $a \mapsto y[a]$ is a first-order definable automorphism of $\tilde{\mathbf{F}}_p$ with parameters in $\mathbf{F}_p$. It will then follow that $y[a] = a^{p^n}$ for some $n \in \mathbf{Z}$. Finally, we show that this implies $y - x^{p^{-n}}$ is a constant.

For $x \in K_0$ let P_x be the set of all poles (finite and at infinity) $\mathfrak{p}$ of x. For each $\mathfrak{p}$ let $\tilde{\mathbf{F}}_p((t_\mathfrak{p}))$ be the completion of K_0 with respect to the valuation $V_\mathfrak{p}$ determined by $\mathfrak{p}$. K_0 is embedded in $\tilde{\mathbf{F}}_p((t_\mathfrak{p}))$. We will make use of the following fact [**1**, p. 103]: Let $x \in \tilde{\mathbf{F}}_p((t_\mathfrak{p}))$, $x = \sum x_i t_p^i$, then $x \in \tau[\tilde{\mathbf{F}}_p((t_\mathfrak{p}))]$ if and only if the following conditions (c_i) for $i < 0$, $i \not\equiv 0 \pmod p$ are satisfied:

$$(c_i) \qquad \sum_{n \geqslant 0} (x_{ip}n)^{1/p^n} = 0.$$

REMARK. Notice that if $v_\mathfrak{p}(x) \geqslant 0$, then $x \in \tau[\tilde{\mathbf{F}}_p((t_\mathfrak{p}))]$, since the left-hand side of (c_i) is zero.

LEMMA 2. *Suppose* $x,\ y \in K_0$ *and* $\operatorname{Link}(y, x)$. *Then the map* $a \to y[a]$ *is a first order definable (with parameters) automorphism of* $\tilde{\mathbf{F}}_p$.

PROOF. Fix $\mathfrak{p} \in P_x$ and let $a \in \tilde{\mathbf{F}}_p$. There exists $b \in \tilde{\mathbf{F}}_p$ such that $ay - bx \in \tau[K_0]$. In the completion $\tilde{\mathbf{F}}_p((t_{\mathfrak{p}}))$, the finitely many equations (c_i) define a first-order formula $\theta_p(u, w, \bar{c}_p)$, where $\bar{c}_p$ is a sequence of parameters (essentially the coefficients in the principal parts of the Laurent expansions of y and x), such that, for a and b as above $\theta_p(a, b, \bar{c}_p)$ holds.

Define $\psi(u, w) = \bigwedge_{\mathfrak{p} \in P_x} \theta_p(u, w, \bar{c}_p)$. We claim that for any $a \in \tilde{\mathbf{F}}_p$, $b = y[a] \Leftrightarrow \psi(a, b)$. The claim follows if we show that given $a \in \tilde{\mathbf{F}}_p$ there exist one and only one $b \in \tilde{\mathbf{F}}_p$ with $\psi(a, b)$. So suppose $\psi(a, b)$ and $\psi(a, b')$. This implies that $(b - b')x \in \tau[\tilde{\mathbf{F}}_p((t_{\mathfrak{p}}))]$ for all $\mathfrak{p} \in P_x$, hence $(b - b')x \in \tau[K_0]$ and since x is regular, $b = b'$. For the existence, take $b = y[a]$.

LEMMA 3. *Let F be an algebraically closed field of characteristic p. Then*

(i) *if* ϕ: $F \to F$ *is an automorphism of F definable over F, there exists* $r(x) \in F(x)$ *and* $n \in \mathbf{N}$ *such that, except for finitely many elements of F, we have* $\phi(a) = r(a)^{1/p^n}$.

(ii) *if* ϕ: $F \to F$ *is an automorphism of F and* $r(x) \in F(x)$ *is such that* $r(a) = \phi(a)$ *for all* $a \in F$ *except for a finite set, then* $r(x) = x^{p^k}$ *for some* $k \in \mathbf{Z}$.

PROOF. Let t be transcendental over F and let $\widetilde{F(t)}$ be the algebraic closure of $F(t)$. Then $F \prec \widetilde{F(t)}$. Let $\tilde{\phi}$ be the definable extension of ϕ to $\widetilde{F(t)}$. We have $\tilde{\phi} \in \operatorname{Aut} \widetilde{F(t)}$.

Let $G = \operatorname{Gal}(\widetilde{F(t)}/F(t))$, $\psi(x, y, \bar{c})$ be a definition of ϕ over F. Then for any $\sigma \in G$ we have

$$\begin{aligned} \widetilde{F(t)} &\models \psi(t, \tilde{\phi}(t), \bar{c}) \\ \Rightarrow \widetilde{F(t)} &\models \psi(\sigma(t), \sigma(\tilde{\phi}(t)), \sigma(\bar{c})) \\ \Rightarrow \widetilde{F(t)} &\models \psi(t, \sigma(\tilde{\phi}(t)), \bar{c}). \end{aligned}$$

Hence $\sigma(\tilde{\phi}(t)) = \tilde{\phi}(t)$, i.e. $\tilde{\phi}(t) \in F(t)^{1/p^\infty}$. This proves $\tilde{\phi}(t) = r(t)^{1/p^n}$ for some $r(x) \in F(x)$ and $n \in \mathbf{N}$. Next we show that the set $A := \{a \in F \mid \tilde{\phi}(a) = r(a)^{1/p^n}\}$ has finite complement. This follows from the following two facts:

(a) A is infinite,

(b) F is strongly minimal.

To prove (a) note that for any finite set $S \subset F$

$$\widetilde{F(t)} \models \exists x\left(\tilde{\phi}(x) = r(x)^{1/p^n} \,\&\, x \notin S\right),$$

hence $F \models \exists x(\tilde{\phi}(x) = r(x)^{1/p^n} \,\&\, x \notin S)$.

To prove (ii) note that for t transcendental (as above) we have infinitely many $a \in F$ such that

$$r(at) = r(a)r(t).$$

Hence the set of zeros Z and the set of poles P of $r(x)$ are closed under multiplication by a for infinitely many $a \in F$. Since Z and P are finite sets we have $Z = P = \{0\} \cup \{\infty\}$, and so $r(x) = x^{p^n}$. □

It follows that $\phi(x) = r(x)$ for all $x \in F$.

COROLLARY. *With the assumptions of Lemma* 3 *there exists* $k \in \mathbf{Z}$ *such that* $y[a] = a^{p^k}$.

The following fact will be used in the proof of Proposition 1.

FACT. If $y,\ x \in K_0$ and Link(y, x), then $P_x = P_y$.

PROOF. By symmetry it is enough to show that $P_y \subseteq P_x$. Suppose not. Let $\mathfrak{p} \in P_y - P_x$. Then, since $v_{\mathfrak{p}}(x) \geqslant 0$, for all $a \in \tilde{\mathbf{F}}_p$ we have $ay \in \tau[\tilde{\mathbf{F}}_p((t_{\mathfrak{p}}))]$. Since $\tilde{\mathbf{F}}_p$ is infinite, it follows by using equation (c_i) that the principal part of the Laurent expansion of y must be zero, hence $v_{\mathfrak{p}}(y) \geqslant 0$, a contradiction.

PROPOSITION 1. *Let* $x,\ y \in K_0$ *and assume that* x *is regular. Then*

$$K_0 \models \mathrm{Link}(y, x) \Leftrightarrow \exists n \in \mathbf{Z} \textit{ and } c \in \tilde{\mathbf{F}}_p \textit{ such that } y = x^{p^n} + c.$$

PROOF. Assume $L(y, x)$. The corollary to Lemma 4 implies that $y[a] = b = a^{p^n}$ for some $n \in \mathbf{Z}$. Hence $ay - a^{p^n}x \in \tau[\tilde{\mathbf{F}}_p((t_{\mathfrak{p}}))]$ for all $\mathfrak{p} \in P_x$ and all $a \in \tilde{\mathbf{F}}_p$. We claim that $v_{\mathfrak{p}}(y - x^{p^{-n}}) \geqslant 0$ for all $\mathfrak{p} \in P_x$. Conditions (c_i) yield the following

$$(c_i) \qquad \sum_{k=0}^{N_0} \left(ay_{ip^k} - a^{p^n}x_{ip^k}\right)^{1/p^k} = 0,$$

where $N_0 \in \mathbf{N}$ and the $y_{ip^k},\ x_{ip^k}$ are the coefficients appearing in the principal part of the Laurent expansions of y and x. Taking p^{N_0}th powers in the above equation gives us a polynomial in a, which is identically zero. By distinguishing the possibilities $N_0 > n,\ N_0 = n$, and $N_0 < n$ one shows that $y_i^{p^n} = x_{ip^n}$ for all $i < 0$.

To conclude the proof note that a pole of $y - x^{p^{-n}}$ is a pole of y or x and by the fact stated before the proposition it is a pole of x. Hence $y - x^{p^{-n}}$ has no poles at all and so must be a constant. For the other implication take $b = a^{p^{-n}}$; uniqueness is a consequence of the regularity of x.

COROLLARY. *For* $y,\ x \in K,\ x$ *regular*,

$$K \models \mathrm{Link}(y, x) \Leftrightarrow \exists n \in \mathbf{Z} \textit{ and } a \in \tilde{\mathbf{F}}_p \textit{ such that } y = x^{p^n} + a.$$

PROOF. Taking p^nth powers yields an automorphism of K. Therefore we may take $y,\ x \in K_0$ and note that regularity over K or K_0 is the same. The result follows from Lemma 1 and Proposition 1.

PROPOSITION 2. $K \models \Phi$.

PROOF. Let $x,\ y \in K$ satisfy Link(y, x). Then $y = x^{p^n} + a$ for some n and a. We may assume $n \neq 0$ since otherwise $y - x$ is a constant. Take $b = a$ and $z = x^{p^{-n}}$. Clearly Link(z, x). Then $x(y - b) = xx^{p^n}$ and $xz = xx^{p^{-n}} = (xx^{p^n})^{p^{-n}}$. We have that $y - z$ is not constant and Link$(xz, x(y - b))$.

2. The interpretation of Φ in L. The set defined by Con(x) in L and in L_0 is the same; denote it by $\tilde{\mathbf{F}}_p^*$. Lemma 1 remains true for the pair $L_0,\ L$. By transfer applied to the structure $(L_0, \mathbf{Z}^*, \tilde{\mathbf{F}}_p^*)$ we get that, for $x,\ y \in L,\ x$ regular,

$$L \models \mathrm{Link}(y, x) \Leftrightarrow \exists n \in \mathbf{Z}^*,\ a \in \tilde{\mathbf{F}}_p^* \text{ such that } y = x^{p^n} + a.$$

PROPOSITION 3. $L \not\models \Phi$.

PROOF. Take $x = t$ and $y = t^{p^N}$ with N infinite and positive. Clearly x is regular and linked to y. Next, for any $a \in \tilde{\mathbf{F}}_p^*$, we claim that $t(t^{p^N} - a)$ is regular. Otherwise, by transfer we could have $bt(t^{p^n} - a') \in \tau[K_0]$ with $b \neq 0$, b, $a' \in \tilde{\mathbf{F}}_p$, and n a positive integer. But this is impossible. So, assume there exists z, $a \in \tilde{\mathbf{F}}_p^*$ such that $\mathrm{Link}(z, t)$ and $\mathrm{Link}(tz, t(t^{p^N} - a))$. Then $z = t^{p^n} + a'$, $a' \in \tilde{\mathbf{F}}_p^*$, $n \in \mathbf{Z}^*$. Notice that if $n < 0$, then n cannot be infinite since $L = \bigcup_{n \in w} L_0^{1/p^n}$.

From $\mathrm{Link}(tz, t(t^{p^N} - a))$ we get an equation of the form

$$t(t^{p^n} + a') = \left(t(t^{p^N} - a)\right)^{p^k} + a'' \tag{1}$$

for some $a'' \in \tilde{\mathbf{F}}_p^*$ and $k \in \mathbf{Z}^*$. Note that a'' must be equal to zero. We show (1) cannot hold.

Case 1. $a' = 0$, (or $a = 0$). Then $a = 0$ and we have $t^{p^n + 1} = t^{p^k} t^{p^{N+k}}$, so $p^n + 1 = p^k + p^{N+k}$. The equation has the following solutions: (a) $k = 0$, $n = N$, and (b) $N + k = 0$, $n = k$. (a) implies $y = z$ which we have excluded and (b) implies $n = -N$ which is also impossible as N is infinite.

Case 2. $a' \neq 0$ and $a \neq 0$. In this case $k = 0$ so $N = n$ and $y - z = a$ which is impossible. Thus Φ is false in L, and $K \not\equiv L$.

I thank G. Cherlin for his help in the preparation of this paper.

REFERENCES

1. G. Cherlin, *Definability in power series rings of nonzero characteristic models and sets*, Lecture Notes in Math., vol. 1103, Springer-Verlag, Berlin and New York, 1984.
2. I. R. Shafarevich, *Basic algebraic geometry*, Springer-Verlag, New York, 1977.
3. L. Van den Dries, *Model theory of fields*, Thesis, Utrecht 1978, Stellingen 4.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903