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A NOTE ON THE EXISTENCE OF G-MAPS BETWEEN SPHERES

STEFAN WANER

ABSTRACT. Let G be a finite group, and let V and W be finite-dimensional real orthogonal G-modules with $V \supset W$, and with unit spheres $S(V)$ and $S(W)$ respectively. The purpose of this note is to give necessary sufficient conditions for the existence of a G-map $f\colon S(V) \to S(W)$ in terms of the Burnside ring of G and its relationship with V and W. Note that if W has a nonzero fixed point, such a G-map always exists, so for nontriviality, we assume this not the case.

Existence of G-maps. Let V be a finite-dimensional orthogonal G-module and let $W \subset V$ be an invariant sub-G-module. Denote the unit spheres of V and W by $S(V)$ and $S(W)$ respectively. Here we obtain an algebraic criterion for the existence of a G-map $f\colon S(V) \to S(W)$. Thus, for nontriviality, we assume $W^G = \{0\}$.

The case $V = W$ has been studied in [3], and we first recall pertinent facts. Let $A(G)$ be the Burnside ring of G. Thus, $A(G)$ is the Grothendieck group of equivalence classes of finite G-sets with addition given by disjoint union. Its elements are thus represented by virtual G-sets, and $A(G)$ is additively the free abelian group with basis $\{G/H\}$, where H runs through representatives of conjugacy classes of subgroups of G. The multiplicative structure is given by cartesian product. One has a natural isomorphism

$$\Phi\colon A(G) \cong \omega_G,$$

where ω_G denotes the zeroth equivariant stable stem. (See, for example, [2]. Roughly, Φ is defined via the collapse map associated with a suitable embedding of a finite G-set in a large sphere $S(V)$.) Denote by $\phi(G)$ the set of conjugacy classes of subgroups of G, and let

$$d\colon A(G) \to \prod_{(H)\in\phi(G)} \mathbf{Z} = C$$

denote its integral closure. Thus $d[s-t]_{(H)} = |s|^H - |t|^H$ for a virtual G-set $s - t$. It is well known that d is a monomorphism [1]. Denote by $\Lambda(W)$ the monoid of (free) G-homotopy classes of G-maps $S(W) \to S(W)$, and let $\nu(W)\colon \Lambda(W) \to A(G)$ denote the natural monoid homomorphisms obtained by suspending and applying Φ^{-1}. The results of [3] give a characterization of the image of $\nu(W)$, which we now state. (The constructions there of G-maps $S(W) \to S(W)$ representing suitable elements in $A(G)$ are given in terms of appropriate tangent G-vector fields on $S(W)$.)

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PROPOSITION. *An element $a = [s - t] \in A(G)$ is in the image of $\nu(W)$ iff the following conditions hold on $s - t$.*

(i) *Recalling that $W^G = \{0\}$, one requires that $s - t$ be the form $1 + \Sigma_i n_i G/H_i$, where the H_i are isotropy subgroups of points in $W - \{0\}$.*

(ii) *If H is an isotropy subgroup in $W - \{0\}$ and $\dim W^H = 1$, then*

$$d(a)_H = \begin{cases} 1 \text{ or } -1 & \text{if } NH \neq H; \\ 0, 1 \text{ or } -1 & \text{if } NH = H. \end{cases} \qquad \square$$

One now has the following

THEOREM. *With V and W as above, there exists a G-map $S(V) \to S(W)$ iff:*

(a) *For each $H \subset G$, $\dim V^H \geqslant 1$ implies $\dim W^H \geqslant 1$.*

(b) *There exists $a \in A(G)$ of the form $1 + \Sigma_i n_i G/H_i$ with each H_i an isotropy subgroup of W such that*

(i) *$d(a)_H = 0$ whenever $\dim V^H > \dim W^H$;*

(ii) *if $\dim V^H = 1$, then*

$$d(a)_H = \begin{cases} 1 \text{ or } -1 & \text{if } NH \neq H; \\ 0, 1 \text{ or } -1 & \text{if } NH = H. \end{cases}$$

(*Note that* (b) *is equivalent to the following assertion*:

(b)′ *There exists an element $a \in \operatorname{Im} \nu(W)$ with $d(a)_H = 0$ whenever $\dim V^H > \dim W^H$.*)

PROOF. We first show that the conditions are necessary. Condition (a) is clearly necessary, while, given any G-map f: $S(V) \to S(W)$, composing with the inclusion i: $S(W) \to S(V)$ gives a G-map g on $S(W)$ whose fixed set degrees are zero whenever $\dim V^H > \dim W^H$, and we take a as $\nu(W)(g)$.

Conversely, assume that conditions (a) and (b) hold. Choose $a \in \operatorname{Im} \nu(W)$ satisfying condition (b), and choose a G-map ρ: $S(W) \to S(W)$ with $\nu(W)(\rho) = a$. By condition (a), if $H \subset G$, is such that $\dim V^H = 1$, then $\dim W^H = 1$ as well. This, together with condition (a) itself, permits one to define a G-map λ_0 from the zero skeleton of $S(V)$ to $S(W)$, with respect to some G-CW decomposition of $S(V)$. Thus assume that we have constructed a G-map

$$\lambda_n \colon S(V)^n \to S(W),$$

where $S(V)^n$ denotes the n-skeleton of $S(V)$. The obstruction to extending λ_n over a typical $(n + 1) - G$-cell of the form $G/H \times D^{n+1}$ defines an element x of $\pi_n(S(W)^H)$. We consider two cases. If $n < \dim W^H - 1$, then the obstruction vanishes for dimensional reasons, and one may extend over the given G-cell. If $n \geqslant \dim W^H - 1$, then one has $\dim V^H > \dim W^H$. Let $\lambda'_n = \rho \circ \lambda_n$. Then, since now $\deg(\rho^H) = 0$, and since the obstruction to extending λ'_n over the cell is given by $\rho^H_*(x) = 0$, the obstruction now vanishes. Continuing this process inductively now gives the desired result. $\square$

REMARKS. Conditions (a) and (b) always hold in the following situation. Let G be any nonsolvable group. Then one has, by [**1**], a nontrivial idempotent e in $A(G)$, and we may assume that $d(e)_{\{1\}} = 0$, where $\{1\}$ denotes the trivial subgroup of G.

If H is a minimal subgroup for which $d(e)_H = 1$, then, if $\not{f}$: $A(G) \to A(H)$ denotes the forgetful homomorphism (which assigns to any virtual G-set the associated H-set via restriction), one has, in $A(H)$, $d(\not{f}(e))_K = 0$ for all proper subgroups $K \subset H$, while $d(\not{f}(e))_H = 1$. Let R be the reduced regular representation of H, let $W = R \oplus R$, and let $V = W \oplus W$. Then $\nu(W)$ contains all H-sets of the form $1 + \sum_i H/K_i$ wiht $K_i \subset H$ proper, whence it contains $\not{f}(e)$. It follows from the theorem that there is an H-map $S(V) \to S(W)$. This in turn gives a G-map $S(iV) \to S(iW)$, where i denotes induction.

The existence of such G-maps is by no means restricted to nonsolvable, or even to nonabelian groups; let $G = \mathbf{Z}/p \times \mathbf{Z}/q$, with p and q distinct primes. Choose integers m and n with $mp + nq = 1$, and let $V = \rho_p \oplus \rho_q \oplus \rho_{pq}$, $W = \rho_p \oplus \rho_q$, where ρ_p is any one-dimensional irreducible complex $\mathbf{Z}/p$-module, regarded as a $(\mathbf{Z}/p \times \mathbf{Z}/q \cong \mathbf{Z}/pq)$-module via projection, and similarly for ρ_q and ρ_{pq}. Then $S(V)$ and $S(W)$ possess isomorphic fixed-sets by any nontrivial subgroup, and we may take $a = 1 - m\mathbf{Z}/p - n\mathbf{Z}/q$ as our element in $A(\mathbf{Z}/pq)$.

We state an easy consequence of the theorem.

COROLLARY. *Let $W \subset V$ be any G-modules with $V^G = W^G$, and assume that if $H \subset G$ and $V^H \neq V^G$, then $W^H \neq W^G$. Denote the orthogonal complement of V^G by $V(G)$, and similarly for $W(G)$. Then there exists a G-map f: $S(V) \to S(W)$ with fixed-set degree prime to $|G|$ iff* (b) *above holds with V and W there replaced by $V(G)$ and $W(G)$ respectively.* (*The condition on fixed sets by subgroups may be thought of as a mild "gap hypothesis," and guarantees that* (a) *holds in this context.*)

PROOF. Note that if $V^G = W^G = 0$, then this is just a restatement of the theorem. Thus assume $\dim V^G \geqslant 1$. Conditions (a) and (b) are certainly sufficient; one may suspend any G-map $S^n \to S^n$, where $n = \dim V^G - 1$, with the unreduced suspension of a G-map $S(V(G)) \to S(W(G))$ to obtain a G-map of the desired degree. Conversely, given a G-map f: $S(V) \to S(W)$ with fixed-set degree prime to $|G|$, one has, for suitable m, $m\deg(f^G) \equiv 1 \bmod |G|$ (and where we may take $m = \pm 1$ if $\dim V^G = 1$). This in turn gives an element $a \in A(G)$ satisfying condition (b) of the hypothesis of the theorem. Indeed, one obtains, by classical general position arguments, an element $a \in A(G)$ which represents the G-map $mf \circ$(inclusion of $S(W)$ in $S(V)$). The crucial point here is that, since $V^G \neq 0$, general position arguments work, since we have a stationary basepoint to map into. Observing that any representing virtual G-set has orbit-types those of W, and that if $H \neq G$ is any isotropy subgroup occurring in W, one has $\dim W^H \neq 1$, one now sees that $a \in \nu(W(G))$ and satisfies the conditions (a) and (b) of the theorem. □

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DEPARTMENT OF MATHEMATICS, HOFSTRA UNIVERSITY, HEMPSTEAD, NEW YORK 11550