PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 1, January 1987

SHORTER NOTES

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AN ESTIMATE FOR THE VARIANCE OF A BOUNDED MEASURABLE RANDOM VARIABLE

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ABSTRACT. An estimate is provided for the variance of a real-valued essentially bounded measurable random variable in terms of its ess sup and ess inf.

Let $I^n = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : 0 \le x_i \le 1,\ i = 1, 2, \dots, n\}$ and let μ be the usual Lebesgue measure on $\mathbf{R}^n$. Let $L_\infty(I^n)$ denote the space of all real-valued, measurable, and essentially bounded functions on I^n.

Our main result is the following:

THEOREM. (a) $\forall f \in L_\infty(I^n)$,

$$\int_{I^n} f^2\,d\mu - \left(\int_{I^n} f d\mu\right)^2 \le \left(\frac{B-b}{2}\right)^2, \tag{1}$$

where $B = \operatorname{ess\,sup} f$ *and* $b = \operatorname{ess\,inf} f$.

(b) *The estimate is the best possible, i.e. there exist* $f \in L_\infty(I^n)$ *for which equality holds.*

PROOF. In any Hilbert space $\mathcal{H}$ over $\mathbf{R}$, we have for u, $v \in \mathcal{H}$, $|v| = 1$,

$$0 \le |u|^2 - (u,v)^2 = |u - (u,v)v|^2 = \min_{\lambda \in \mathbf{R}} |u - \lambda v|^2. \tag{2}$$

Now let $\mathcal{H} = L_2(I^n)$ and $v = 1$. Then, for $f \in L_2(I^n)$, (2) reduces to

$$\|f\|^2 - (f,1)^2 = \min_{\lambda \in \mathbf{R}} \int_{I^n} \{f(x) - \lambda\}^2 d\mu. \tag{3}$$

For all $f \in L_\infty(I^n)$,

$$\left| f(x) - \frac{B+b}{2} \right| \le \frac{B-b}{2} \quad \text{a.e.} \tag{4}$$

Since $L_\infty(I^n) \subset L_2(I^n)$, combining (3) and (4) we obtain

$$\|f\|^2 - (f,1)^2 \le \int_{I^n} \left\{ f(x) - \frac{B+b}{2} \right\}^2 d\mu \le \left(\frac{B-b}{2}\right)^2.$$

Received by the editors January 13, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 41A17; Secondary 46E20.

This proves (a). To prove (b), for a given B and b, and for $x \in I^n$, define f as follows:

$$f(x_1, \dots, x_n) = \begin{cases} B & \text{if } 0 \le x_1 \le \frac{1}{2}, \\ b & \text{if } \frac{1}{2} < x_1 \le 1. \end{cases} \tag{5}$$

Then, ess sup $f = B$, ess inf $f = b$, and

$$\int_{I^n} f^2 d\mu - \left(\int_{I^n} f d\mu\right)^2 = \frac{B^2 + b^2}{2} - \left(\frac{B+b}{2}\right)^2 = \left(\frac{B-b}{2}\right)^2$$

so that equality holds in (1) for this $f \in L_\infty(I^n)$. □

For $f \in C(I^n)$ the estimate provided in (1) becomes

$$\int_{I^n} f(x)^2 dx - \left(\int_{I^n} f(x) dx\right)^2 \le \left(\frac{M-m}{2}\right)^2, \tag{6}$$

where M and m are the (absolute) maximum and minimum of f. That the estimate (6) is the best possible in $C(I^n)$ follows from part (b) of the above theorem and the fact that $C(I^n)$ is dense in $L_\infty(I^n)$. As an example, for $0 < \varepsilon < \frac{1}{2}$, consider $f_\varepsilon \in C(I^n)$ defined by

$$f_\varepsilon(x_1, \dots, x_n) = \begin{cases} M, & 0 \le x_1 \le \frac{1}{2} - \varepsilon, \\ \dfrac{M+m}{2} + \dfrac{M-m}{2}\left(\dfrac{\frac{1}{2} - x_1}{\varepsilon}\right), & \frac{1}{2} - \varepsilon \le x_1 \le \frac{1}{2} + \varepsilon, \\ m, & \frac{1}{2} + \varepsilon \le x_1 \le 1. \end{cases} \tag{7}$$

For this $f_\varepsilon(x)$ it can be shown that

$$\int_{I^n} f_\varepsilon(x)^2 dx - \left(\int_{I^n} f_\varepsilon(x) dx\right)^2 = \left(\frac{M-m}{2}\right)^2 \left(1 - \frac{4}{3}\varepsilon\right), \tag{8}$$

which illustrates that the bound in (6) is the best possible in $C(I^n)$.

Finally we remark that the estimate given in (6) for the particular case $n = 1$ solves the problem posed in [**1**].

We are grateful to the referee for his valuable suggestions.

References

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