

ON THE IDENTITY $L(E, F) = LB(E, F)$
FOR PAIRS OF LOCALLY CONVEX SPACES E AND F

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ABSTRACT. Some new pairs (E, F) of locally convex spaces are given such that every continuous linear mapping from E into F is bounded.

The aim of this note is to study pairs of locally convex spaces (l.c.s.) (E, F) such that every continuous linear mapping from E into F is bounded. We mainly concentrate on the cases when E or F coincide with a countable product ω or a countable direct sum φ of copies of the scalar field \mathbf{K} ($\mathbf{K} = \mathbf{R}$ or \mathbf{C}). In these cases complete characterizations are provided. Some of the conditions which occur in our study have been extensively considered in the literature with different purposes (see [4, 8, 10, 11, 12, 19]).

We shall use standard notations of locally convex spaces as in [14 and 15]. If E is a l.c.s., the set of all continuous seminorms on E will be denoted by $cs(E)$. If E and F are l.c.s., $L(E, F)$ denotes the linear space of all continuous linear mappings from E into F . A linear mapping $f: E \rightarrow F$ is said to be *bounded* if there is a 0-neighborhood U in E such that $f(U)$ is bounded in F . We write $LB(E, F)$ for the set of all linear bounded mappings from E into F . Clearly $LB(E, F) \subset L(E, F)$. Following Floret [12], a l.c.s. is said to satisfy the *countable neighborhood property* if for every sequence (p_n) in $cs(E)$ there are $c_n > 0$ and $p \in cs(E)$ such that $p_n \leq c_n p$, $n \in N$. This condition already appears in [19]. Every (gDF) -space has the c.n.p. (see [14, Chapter 12]). More information about the c.n.p. can be seen in [4 and 12].

A pair (E, F) of l.c.s. is said to satisfy the *localization property* [5] if every equicontinuous subset A of $L(E, F)$ is equibounded (i.e., there is a 0-neighborhood U in E such that $\bigcup\{f(U): f \in A\}$ is bounded in F). Clearly if (E, F) has the localization property, then $L(E, F) = LB(E, F)$. Each of the following pairs has the localization property (see [5, Proposition 4]):

- (a) E or F is a normed space.
- (b) E has the c.n.p. and F is metrizable.
- (c) E is metrizable and F has a countable basis of bounded sets.

The localization property for pairs of Fréchet spaces was characterized by Vogt in [21]. By [21, 1.2], if (E, F) is a pair of Fréchet spaces, then $L(E, F) = LB(E, F)$ if and only if (E, F) has the localization property. Defant observed that Vogt's ideas in [21] can be used to show that this equivalence remains true if E and F are (DF) -spaces and F is locally complete [14]. Applications of the localization property can be seen in [5, 6, 7, 16, 21, and 22].

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A barrelled l.c.s. is said to be *quasi-Baire* [18] if it cannot be covered by an increasing sequence of rare subspaces, or, equivalently, if it does not contain a complemented copy of φ (see [2]).

PROPOSITION 1. *Let E be a barrelled space and let $F = \text{ind } F_n$ be a strict inductive limit of Banach spaces. The following are equivalent (tfae):*

- (1) (E, F) has the localization property.
- (2) $L(E, F) = LB(E, F)$.
- (3) E is quasi-Baire.

PROOF. (3) \rightarrow (1). Let A be an equicontinuous subset of $L(E, F)$. For each $n \in N$ we set $E_n := \{x \in E \mid f(x) \in F_n \text{ for each } f \in A\}$. Clearly (E_n) is an increasing sequence of closed subspaces of E . Given $x \in E$, the set $A(x) := \{f(x) \mid f \in A\}$ is bounded in F , hence there is a certain s with $A(x) \subset F_s$, therefore $x \in E_s$. Since E is quasi-Baire, there is a certain m with $E_m = E$. Thus each $f \in A$ factors through F_m . Since F_m is a Banach subspace of F , the conclusion follows from result (a) in the introduction.

(2) \rightarrow (3). If E is not quasi-Baire, there is a continuous linear mapping g from E onto φ [2]. Since F is strict, there is an isomorphism h from φ into F such that $h(\varphi)$ is not included in F_n for each $n \in N$. Define $f: E \rightarrow F$ by $f := h \circ g$. Clearly $f \in L(E, F)$ but it is not bounded, a contradiction. \square

REMARK 2. (3) \rightarrow (1) above does not depend on the barrelledness of E , but if $E_0 := (\varphi, \sigma(\varphi, \omega))$, then $L(E_0, \varphi) = LB(E_0, \varphi)$ [8, 2.12(b)], and E_0 is a countable increasing union of rare subspaces.

A sequence (x_n) in a l.c.s. E is said to be *very strongly convergent* if given any element (c_n) of ω , then $(c_n x_n)$ tends to 0 in E , or equivalently if $(p(x_n)) \in \varphi$ for every $p \in \text{cs}(E)$ (see [10, 2.50]). A l.c.s. is said to satisfy the *vanishing sequence property* (v.s.p.) if given any strongly convergent sequence (x_n) there is $m \in N$ such that $x_n = 0$ if $n > m$. This condition already appears in [19] in a study of functions whose support is "scalarly compact". Clearly every l.c.s. with a continuous norm and every l.c.s. with the c.n.p. has the v.s.p. A Fréchet space satisfies the v.s.p. if and only if it has a continuous norm (see [11]). Applications of l.c.s. with the v.s.p. to infinite dimensional holomorphy can be seen in [10 and 11]. A study of l.c.s. with the v.s.p. is included in [20].

Following Bellenot and Dubinsky, [1], a Fréchet space E is called a *quojection* if for every $p \in \text{cs}(E)$ we have that $E/\ker p$ is normable. These Fréchet spaces are precisely the countable surjective limits of Banach spaces, [10, Chapter 6]. They have been considered by several authors (see e.g. [1, 9, 11, 13]). Every countable product of Banach spaces is a quojection. Moscatelli, [17], gave examples of quojections which are not isomorphic to a product of Banach spaces.

LEMMA 3. *Let E be a nonnormable Fréchet space and let F be a locally complete l.c.s. If $L(E, F) = LB(E, F)$, then F has the v.s.p.*

PROOF. If F does not have the v.s.p., then it contains a subspace G isomorphic to ω by [20, Theorem 1] (which also holds for F locally complete). Now it is well known that E has a quotient H isomorphic to ω . Thus, if $Q: E \rightarrow H$ is the quotient mapping and $f: H \rightarrow G$ is an isomorphism, then $f \circ Q \in L(E, F)$ and it is not bounded, a contradiction. \square

The converse of the former lemma is not true, as shown by the identity mapping of a Fréchet space with a continuous norm.

PROPOSITION 4. *Let E be a quojection which is not normable and let F be a locally complete l.c.s. Tfae:*

- (1) (E, F) has the localization property.
- (2) $L(E, F) = LB(E, F)$.
- (3) F has the v.s.p.
- (4) F does not contain a complemented copy of ω .

PROOF. (1) \rightarrow (2) is obvious and (2) \rightarrow (3) follows from Lemma 3. (3) and (4) are equivalent according to [20, Theorem 1]. It remains to prove that (3) implies (1). Let (q_n) be an increasing fundamental sequence of continuous seminorms on E . Let A be an equicontinuous subset of $L(E, F)$. We claim the existence of $m \in N$ such that $f(x) = 0$ for every $f \in A$ and $x \in E$ with $q_m(x) = 0$. Assume the contrary. We can select a sequence (x_m) in E with $q_m(x_m) = 0$ for each m , and a sequence (f_m) in A such that $f_m(x_m) \neq 0$ for every $m \in N$. Take $p \in cs(F)$. Since A is equicontinuous, there are $k > 0$ and $s \in N$ such that $p(f(y)) \leq kq_s(y)$ for every $y \in E$ and $f \in A$. Hence, if $m > s$, we have that $0 \leq p(f_m(x_m)) \leq kq_s(x_m) \leq kq_m(x_m) = 0$. Therefore $(f_m(x_m))$ is a nontrivial very strongly convergent sequence in F , contradicting the vanishing sequence property.

Let m be the positive integer determined in the claim. Given $f \in A$ there is a unique continuous linear mapping \tilde{f} from $E/\ker q_m$ into F such that $f = \tilde{f} \circ Q$, $Q: E \rightarrow E/\ker q_m$ being the canonical surjection. Set $\tilde{A} := \{\tilde{f} | f \in A\}$. Clearly \tilde{A} is an equicontinuous subset of $L(E/\ker q_m, F)$. By assumption $E/\ker q_m$ is normable. We denote by V the closed unit ball of the norm defining the topology of $E/\ker q_m$. Therefore $\bigcup(\tilde{f}(V) | \tilde{f} \in \tilde{A})$ is bounded in F , thus $U := Q^{-1}(V)$ is a 0-neighborhood in E such that $\bigcup(f(U) | f \in A)$ is bounded in F . \square

REMARK 5. *For $E = \omega$, (1)–(4) of Proposition 4 are also equivalent to (5) if $f \in L(\omega, F)$, then $\dim f(\omega)$ is finite. Indeed, it is shown in [20, Theorem 1] that if $f \in L(\omega, F)$, then either $\dim f(\omega)$ is finite or $f(\omega)$ is isomorphic to ω . Clearly, in the first case f is bounded, while in the second f is not, since the induced mapping $f_0: \omega/\ker f \rightarrow f(\omega)$ is an isomorphism.*

REMARK 6. The local completeness assumption is not needed to prove (3) \rightarrow (1) in Proposition 4. In fact, if E is a quojection and F is a l.c.s. with the v.s.p., then (E, F) has the localization property. But observe that $F_0 := \varphi$ endowed with the topology induced by ω is a nonlocally complete l.c.s. without v.s.p. but each $f \in L(\omega, F_0)$ is bounded, since its range is finite dimensional [8, 2.15(b)].

Proposition 4 is, in some sense, “dual” to Proposition 1. The strong dual of a strict (LB) -space is a quojection and the strong dual of a nonnormable quojection is a strict (LB) -space (see [9]). Our next characterization emphasizes the aforementioned “duality” and, in particular, implies that the v.s.p. is a property of the dual pair.

PROPOSITION 7. *Let F be a l.c.s. Tfae:*

- (1) F has the vanishing sequence property.
- (2) If F' is the increasing union of a sequence (G_n) of subspaces of F' , then there is $m \in N$ such that G_m is dense in $(F', \sigma(F', F))$.

PROOF. (1) \rightarrow (2). Suppose that F' is the increasing union of a sequence (G_n) of $\sigma(F', F)$ -closed subspaces such that $G_n \neq G_{n+1}$ for each $n \in N$. Take $u_n \in G_{n+1} \setminus G_n$, $n \in N$. By Hahn-Banach's theorem we can select $x_n \in F$ such that $u_n(x_n) = 1$, $u(x_n) = 0$ for each $u \in G_n$, $n \in N$. If $p \in \text{cs}(E)$, the polar set U° of $U := \{x \in E | p(x) \leq 1\}$ is a Banach disc in $(F', \sigma(F', F))$, hence there is $s \in N$ such that $U^\circ \subset G_s$. Therefore $v(x_n) = 0$ for each $v \in U^\circ$ and $n > s$. This implies that $p(x_n) = 0$ if $n > s$. Now, by (1), there is some $m \in N$ with $x_n = 0$ if $n > m$. This contradicts that $u_n(x_n) = 1$ for every $n \in N$.

(2) \rightarrow (1). Let (y_n) be a sequence in F such that $(p(y_n)) \in \varphi$ for every $p \in \text{cs}(F)$. For each $n \in N$, let G_n denote the orthogonal in F' of $\{y_n, y_{n+1}, \dots\}$. Clearly (G_n) is an increasing sequence of closed subspaces of $(F', \sigma(F', F))$. Given any $u \in F'$, $(u(y_n)) \in \varphi$, then $u \in G_s$ for a certain s and $F' = \bigcup_{n=1}^\infty G_n$. By (2), there is $m \in N$ with $G_m = F'$. This implies $y_n = 0$ if $n > m$. \square

Bellenot and Dubinsky, [1], introduce the following condition $(*)$ on a Fréchet space E : E' is not the union of an increasing sequence of Banach spaces E'_n with E'_n being a closed subspace of E'_{n+1} . They prove that every quojection does not satisfy condition $(*)$. The converse being true if E is reflexive. In our next theorem \mathcal{NK} denotes the class of all nuclear Köthe spaces with a continuous norm.

THEOREM 8. Let E be a Fréchet space. Tfae:

- (1) $(E, \lambda(A))$ has the localization property for every $\lambda(A) \in \mathcal{NK}$.
- (2) $L(E, \lambda(A)) = LB(E, \lambda(A))$ for every $\lambda(A) \in \mathcal{NK}$.
- (3) $E \otimes_p i\lambda(A)'_b$ is barrelled for every $\lambda(A) \in \mathcal{NK}$.
- (4) E does not satisfy condition $(*)$ of Bellenot and Dubinsky.

PROOF. The equivalence of (1), (2) and (3) is a consequence of [21, 1.2 and 7.3], and (3) and (4) are equivalent by [3, Theorem 2 and Remark below]. \square

REMARK 9. The class \mathcal{NK} can be replaced in Theorem 8 by the class of reflexive Fréchet spaces with the bounded approximation property and a continuous norm (compare with our Proposition 15 and Corollary 16).

COROLLARY 10. Let E be a reflexive Fréchet space. Tfae:

- (1) (E, F) has the localization property for every l.c.s. F with the v.s.p.
- (2) $L(E, F) = LB(E, F)$ for every l.c.s. F with the v.s.p.
- (3) E is a quojection.

PROOF. (3) \rightarrow (1) is a consequence of Remark 6. (2) \rightarrow (3) follows from Theorem 8, since E is reflexive. \square

A l.c.s. is said to satisfy the *countable boundedness condition* (c.b.c.) if given a sequence (B_n) of bounded subsets of F , there are $c_n > 0$, $n \in N$, such that $\bigcup_{n=1}^\infty c_n B_n$ is bounded. A l.c.s. is said to satisfy the *individual countable boundedness condition* (i.c.b.c.) if given any sequence (y_n) in F , there are $c_n > 0$, $n \in N$, such that $(c_n y_n)$ is bounded, or, in other words, if every sequence in F is very weakly convergent, [10, 2.51]. Every metrizable space satisfies the c.b.c. These conditions have been used by S. Dierolf [8] in a study of the commutability of inductive and projective limits and spaces of continuous linear mappings. She observes [8, p. 27], that the subspace of $\mathbf{K}^{\mathbf{R}}$ of all the elements with countably many coordinates distinct from 0, endowed with the topology induced by $\mathbf{K}^{\mathbf{R}}$, has the i.c.b.c. but not the c.b.c. Applications of l.c.s. with the i.c.b.c. to infinite holomorphy can be seen in [11].

PROPOSITION 11. *Let F be a l.c.s. Tfae:*

- (1) F satisfies the c.b.c.
- (2) (φ, F) has the localization property.

PROOF. (1) \rightarrow (2) is a particular case of [8, 2.6(a)].

(2) \rightarrow (1). Let (B_n) be a sequence of bounded subsets of F . We put $I := \prod_{n=1}^{\infty} B_n$. For each $i \in I$, $i = (i_n)$, $i_n \in B_n$, $n \in N$, we define $f_i: \varphi \rightarrow F$ by $f_i(a) := \sum_{n=1}^{\infty} a_n i_n$, which is a well-defined continuous linear mapping. Let U be an absolutely convex 0-neighborhood in F . If we show that $V := \bigcap (f_i^{-1}(U) | i \in I)$ is absorbent in φ , then $(f_i | i \in I)$ is an equicontinuous subset of $L(\varphi, F)$. To prove that V is absorbent it is enough to show that each e_m is absorbed by V , e_m being the canonical unit vector in φ . Take $m \in N$. If $i \in I$, $i = (i_n)$, $f_i(e_m) = i_m$, therefore $(f_i(e_m) | i \in I) = B_m$, which is bounded in F , hence there is $b_m > 0$ such that $f_i(e_m) \in b_m U$ for every $i \in I$. Thus $e_m \in b_m V$. Now, $(f_i | i \in I)$ being equicontinuous, we apply (2) to obtain that it is equibounded, hence there is a 0-neighborhood W in φ such that $B := \bigcup (f_i(W) | i \in I)$ is bounded in F . Given $n \in N$, there is $c_n > 0$ with $e_n \in c_n W$. Then $f_i(c_n^{-1} e_n) \in B$ for each $i \in I$, hence $i_n \in c_n B$ for each $i_n \in B_n$. Thus $c_n^{-1} B_n \subset B$ for every $n \in N$ and F satisfies the c.b.c. \square

COROLLARY 12. *Let E be a barrelled (DF)-space and F a l.c.s. Consider the following statements:*

- (1) F satisfies the c.b.c.
- (2) (E, F) has the localization property.

Then, (a) (1) always implies (2), (b) if E does not contain a total bounded subset, then (1) and (2) are equivalent.

PROOF. (a) is a consequence of [8, 2.6(a)]. To prove (b) assume that F does not satisfy the c.b.c. By Proposition 11, there is an equicontinuous subset $A \subset L(\varphi, F)$ which is not equibounded. As E does not contain a total bounded subset, there is a continuous linear mapping g from E onto φ [2]. Then $A := \{f \circ g | f \in A\}$ is an equicontinuous subset of $L(E, F)$ which is not equibounded, a contradiction. \square

PROPOSITION 13. *Let F be a l.c.s. Tfae:*

- (1) F satisfies the i.c.b.c.
- (2) $L(\varphi, F) = LB(\varphi, F)$.

PROOF. It is similar to the one of Proposition 11, but easier. \square

A l.c.s. is said to satisfy the *countable linear form property (c.l.f.p.)* if given any sequence (u_n) in E' , there is a 0-neighborhood U in E such that u_n belongs to the linear span of U° for every $n \in N$. It is easy to see that the c.n.p. implies the c.l.f.p. The converse is false as an analysis of our example [4, 1.7] shows.

PROPOSITION 14. *Let E be a l.c.s. Tfae:*

- (1) E has the c.l.f.p.
- (2) $L(E, \omega) = LB(E, \omega)$.

PROPOSITION 15. *Let E be a l.c.s. Tfae:*

- (1) E has the c.n.p.
- (2) (E, ω) has the localization property.

PROOF. (1) \rightarrow (2) follows from result (b) in the introduction [5].

(2) \rightarrow (1). Let (U_n) be a sequence of absolutely convex closed 0-neighborhoods in E . We set $B_n := U_n^\circ$ and $I := \prod_{n=1}^\infty B_n$. If $i \in I$, $i = (i_n)$, $i_n \in B_n$, $n \in N$, we define $f_i: E \rightarrow \omega$ by $f_i(x) := (i_n(x))$, $x \in E$. Clearly $f_i \in L(E, \omega)$ for every $i \in I$. We show that $(f_i | i \in I)$ is an equicontinuous subset of $L(E, \omega)$. If V is a 0-neighborhood in ω , there are $m \in N$ and $c_n > 0$, $n = 1, \dots, m$, such that $W := \{a \in \omega \mid |a_n| \leq c_n, n = 1, \dots, m\} \subset V$. Clearly $M := \bigcap_{n=1}^m c_n U_n$ is a 0-neighborhood in E and if $x \in M$ and $i = (i_n) \in I$ we have that $f_i(x) = (i_n(x)) \in W$. By (2), $(f_i | i \in I)$ is equibounded, hence there is a closed absolutely convex 0-neighborhood U in E such that $B := \bigcup (f_i(U) | i \in I)$ is bounded in ω . Therefore for each $n \in N$ there is $b_n > 0$ such that $|\pi_n(f_i(x))| \leq b_n$ for every $x \in U$, π_n being the canonical n th projection from ω onto K . Then for each $n \in N$ and $u \in B_n = U_n^\circ$ we get $|u(x)| \leq b_n$ for every $x \in U$, hence $U_n^\circ \subset b_n U^\circ$ and this implies that U_n contains $b_n^{-1}U$. Thus U is contained in $\bigcap_{n=1}^\infty b_n U_n$ and E satisfies the c.n.p. \square

COROLLARY 16. *Let E be a l.c.s. and F a Fréchet space which does not have a continuous norm. Tfae:*

- (1) E has the c.n.p.
- (2) (E, F) has the localization property.

PROOF. (1) \rightarrow (2) is a consequence of result (b) in the introduction [5].

(2) \rightarrow (1). Assume that E does not satisfy the c.n.p. By Proposition 15, there is an equicontinuous subset A of $L(E, \omega)$ which is not equibounded. Since E does not have a continuous norm, it contains ω as a complemented subspace. Thus we conclude that (E, F) does not have the localization property. \square

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REFERENCES

1. S. Bellenot and E. Dubinsky, *Fréchet spaces with nuclear Köthe quotients*, Trans. Amer. Math. Soc. **273** (1982), 579–594.
2. J. Bonet and P. Pérez Carreras, *Some results on barrelledness in projective tensor products*, Math. Z. **185** (1984), 333–338.
3. J. Bonet, *Quotients and projective tensor products*, Arch. Math. **45** (1985), 169–173.
4. —, *The countable neighborhood property and tensor products*, Proc. Edinburgh Math. Soc. **28** (1985), 207–215.
5. A. Defant and K. Floret, *The precompactness-lemma for sets of operators*, Functional Analysis, Holomorphy and Approximation Theory. II (G. Zapata, Ed.), North-Holland, Amsterdam, 1984, pp. 39–55.
6. —, *Localization and duality of topological tensor products*, Collect. Math. **35** (198-), 43–61.
7. A. Defant and W. Govaerts, *Tensor products and spaces of vector valued continuous functions*, preprint, 1984.
8. S. Dierolf, *On spaces of continuous linear mappings between locally convex spaces*, Habilitationsschrift, Munich, 1984.
9. S. Dierolf and D. N. Zarnadze, *A note on strictly regular Fréchet spaces*, Arch. Math. **42** (1984), 549–556.
10. S. Dineen, *Complex analysis in locally convex spaces*, North-Holland Math. Studies, no. 57, North-Holland, Amsterdam, 1981.
11. —, *Surjective limits of locally convex spaces and their application to infinite dimensional holomorphy*, Bull. Soc. Math. France **103** (1975), 441–509.
12. K. Floret, *Some aspects of the theory of locally convex inductive limits*, Functional Analysis, Surveys and Recent Results. II (K. D. Bierstedt and B. Fuchsteiner, Eds.), North-Holland Math. Studies, no. 38, North-Holland, Amsterdam, 1980, pp. 205–237.

13. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. No. 16 (1955).
14. H. Jarchow, *Locally convex spaces*, Teubner, Stuttgart, 1981.
15. G. Köthe, *Topological vector spaces*, I and II, Springer, 1969 and 1979.
16. R. Meise and D. Vogt, *Holomorphic functions of uniformly bounded type on nuclear Fréchet spaces*, preprint, 1984.
17. V. B. Moscatelli, *Fréchet spaces without continuous norms and without bases*, Bull. London Math. Soc. **12** (1980), 63–66.
18. S. A. Saxon, *Nuclear and product spaces, Baire-like spaces and the strongest locally convex topology*, Math. Ann. **197** (1972), 87–106.
19. L. Schwartz, *Espaces de fonctions différentiables à valeurs vectorielles*, J. Analyse Math. **4** (1954/55), 88–148.
20. M. A. Simoes, *Very strongly and very weakly convergent sequences in locally convex spaces*, Proc. Roy. Irish Acad. Sect. A **84** (1984), 125–132.
21. D. Vogt, *Frécheträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. **345** (1983), 182–200.
22. ———, *Some results on continuous linear maps between Fréchet spaces*, Functional Analysis, Surveys and Recent Results. III (K. D. Bierstedt and B. Fuchssteiner, Eds.), North-Holland Math. Studies, vol. 90, North-Holland, Amsterdam, 1984, pp. 349–381.

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