PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 4, April 1987

UNCONDITIONAL BASES IN $L^2(0,a)$

SERGEĬ V. HRUŠČËV

ABSTRACT. A method is given for producing unconditional bases in subspaces $K_\theta = H^2 \ominus \theta H^2$ of the Hardy space H^2, θ being an inner function in the upper half-plane. For $\theta = \exp(iaz)$ the space K_θ is the Fourier-Laplace transform of $L^2(0,a)$, which allows us to establish a necessary and sufficient condition for certain families of functions (including exponentials) to constitute unconditional bases in $L^2(0,a)$.

1. A family of nonzero vectors $\{e_n\}$ in a Hilbert space H is called an unconditional basis in H if every element x in H can be uniquely decomposed in an unconditionally convergent series

$$x = \sum_n \alpha_n e_n, \qquad \alpha_n \in \mathbf{C}.$$

The Köthe-Toeplitz theorem says that a complete family $\{e_n\}$ is an unconditional basis iff for some c, $0 < c < 1$, the following "approximate Parseval identity" holds:

$$c \cdot \sum_n |\alpha_n|^2 \cdot \|e_n\|^2 \le \left\| \sum_n \alpha_n e_n \right\|^2 \le c^{-1} \sum_n |\lambda_n|^2 \cdot \|e_n\|^2$$

for every finite sequence $\{\alpha_n\}$ of complex numbers.

A classical example of an unconditional basis, in fact orthogonal, is given by the family of exponentials $\{e^{inx}\}_{n\in\mathbf{Z}}$, $H = L^2(0,2\pi)$. Unconditional bases of exponentials $\{e^{i\lambda_n x}\}$ in $L^2(0,a)$ have been described in [**1**] under the assumption $\inf_n \operatorname{Im}\lambda_n > -\infty$ ($\sup_n \operatorname{Im}\lambda_n < +\infty$). The functions w satisfying the Muckenhoupt condition (A_2) on the real line $\mathbf{R}$,

$$\sup_{I\in\mathcal{I}} \left(\frac{1}{|I|} \int_I w\,dx \right) \cdot \left(\frac{1}{|I|} \int_I w^{-1}\,dx \right) < +\infty, \tag{1}$$

$\mathcal{I}$ being the family of finite intervals, play an important role in this description. It was observed later [**2**] that the theory of Hankel operators permits one to extend the result of [**1**] to families of reproducing kernels of the Hardy class H^2 in the upper half-plane $\mathbf{C}_+$ (see [**3**] for details). The methods developed in [**1** and **3**] can be applied to the description of the resonance frequences of semi-infinite strings corresponding to unconditional bases of resonance states [**4**]. One more application

Received by the editors January 30, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46A35, 47B35.
Key words and phrases. Toeplitz operator, Muckenhoupt condition, unconditional bases.
Research partially supported by the Institut D'Estudis Catalans Centre de Recerca Matematica in Barcelona, Spain.

has been considered recently in [**5**] to investigate the unconditional basis property of $\{x^{\beta-1}E_{1,\beta}(ix\lambda_n)\}$ in $L^2(0,a)$. Here

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

is a Mittag-Leffler function [**6**]. In this paper the results of [**5**] are extended to more general families. The advantage of the approach presented is that it makes the arguments more transparent even in the case of exponentials ($\beta=1$).

2. Our main tool is Widom's criterion for invertibility of Toeplitz operators T_φ with unimodular symbols φ. We recall that the Toeplitz operator T_φ with symbol φ, φ being a bounded measurable function on $\mathbf{R}$, ($\varphi\in L^\infty(\mathbf{R})$) is defined by

$$T_\varphi f=\mathbf{P}_+(\varphi f),\qquad f\in H^2.$$

Here $\mathbf{P}_+$ stands for the orthogonal projection in $L^2(\mathbf{R})$ onto H^2. We define the Hilbert transform $\tilde{v}$ of $v\in L^\infty(\mathbf{R})$ as follows:

$$\tilde{v}(x)=\frac{1}{\pi}(\text{v.p.})\int\left\{\frac{1}{x-t}+\frac{t}{1+t^2}\right\}v(t)\,dt.$$

We state Widom's theorem in the form used in [**3**] (see Theorem 5). We say that a unimodulalr function φ is a Helson-Szegö function if T_φ is invertible. We denote by H^∞ the Hardy algebra in $\mathbf{C}_+$.

THEOREM (WIDOM [**7**]). *Let φ be a unimodular function. Then the following are equivalent.*

1. *φ is a Helson-Szegö function.*
2. $\operatorname{dist}_{L^\infty}(\varphi,H^\infty)<1$, $\operatorname{dist}(\overline{\varphi},H^\infty)<1$.
3. *There is an outer function $f\in H^\infty$ such that $\|\varphi-f\|_\infty<1$.*
4. *$\varphi=\exp\{i(\tilde{u}+v+c)\}$, where $u,v\in L^\infty(\mathbf{R})$, $\|v\|_\infty<\pi/2$, $c\in\mathbf{R}$.*
5. *There are a constant λ, $|\lambda|=1$, and an outer function h with $|h|^2$ satisfying the Muckenhoupt condition (A_2) on $\mathbf{R}$ such that*

$$\varphi=\lambda\overline{h}/h.$$

REMARK. Notice that T_φ is left-invertible for a unimodular φ iff $\operatorname{dist}(\varphi,H^\infty)<1$ (see [**3**]).

Widom's theorem, as stated above, provides a simple proof of the following well-known lemma. We put $z=x+iy$, $P_z=\pi^{-1}y\cdot(x^2+y^2)^{-1}$ and denote by P_zf the harmonic extension of f from $\mathbf{R}$ to $\mathbf{C}_+$.

LEMMA 1. *Let h be an outer function in $\mathbf{C}_+$. Then $|h|^2$ satisfies* (1) *on $\mathbf{R}$ if and only if*

$$P_z|h|^2\le\text{const}\cdot|h(z)|^2,\qquad P_z|h|^{-2}\le\text{const}\cdot|h(z)|^{-2}.$$

PROOF. If $|h|^2$ satisfies (1) then $T_{\overline{h}/h}$ is invertible and 2 of Widon's theorem implies $\operatorname{dist}(\overline{h}\cdot h^{-1},H^\infty)<1$. It follows that there exist $f\in H^\infty$ and $q<1$ such that

$$|\,|h(x)|^2-f(x)\cdot h^2(x)|<q|h(x)|^2,\qquad x\in\mathbf{R}.$$

Since $P_z f h^2 = f(z)\cdot h^2(z)$ and $\|f\|_\infty < 2$, we obtain

$$P_z|h|^2 \le q\cdot P_z|h|^2 + 2|h(z)|^2, \quad \text{i.e.} \quad P_z|h|^2 \le 2(1-q)^{-1}\cdot|h(z)|^2.$$

Similarly one can prove the second inequality. The converse is a trivial estimate of the Poisson kernel. □

A sequence $\{\lambda_n\}$ in $\mathbf{C}_+$ is said to be interpolating if for every bounded sequence $\{\alpha_n\}$ there is $f\in H^\infty$ such that $f(\lambda_n)=\alpha_n$. The Carleson theorem [**8**] says that $\{\lambda_n\}$ is interpolating iff

$$\inf_n \prod_{k\ne n}\left|\frac{\lambda_k-\lambda_n}{\lambda_k-\overline{\lambda}_n}\right| > 0. \tag{2}$$

Clearly, (2) implies the Blaschke condition in $\mathbf{C}_+$ which allows one to consider the Blaschke product

$$B=\prod_n \varepsilon_n\frac{z-\lambda_n}{z-\overline{\lambda}_n},$$

where the unimodular constants ε_n are chosen so as to provide the convergence of the product in $\mathbf{C}_+$.

Any inner function θ in $\mathbf{C}_+$ generates a subspace $K_\theta = H^2\ominus\theta H^2$ of H^2. We denote by $\mathcal{P}_\theta$ the orthogonal projection onto K_θ. Clearly $\mathcal{P}_\theta=\theta(I-\mathbf{P}_+)\overline{\theta}$.

THEOREM 1. *Let h be an outer function in $\mathbf{C}_+$ satisfying $|h|^2\in(1)$. θ an inner function in $\mathbf{C}_+$, $\{\lambda_n\}$ a sequence in $\mathbf{C}_+$ such that*

$$\sup_n|\theta(\lambda_n)|<1. \tag{3}$$

Then $\{\mathcal{P}_\theta(h(z-\overline{\lambda}_n)^{-1}\}$ is an unconditional basis in K_θ if and only if

(a) *$\{\lambda_n\}$ is an interpolating sequence; and*

(b) *$\overline{h}\cdot h^{-1}\cdot\overline{B}\theta$ is a Helson-Szegö function, B being the Blaschke product associated with $\{\lambda_n\}$.*

3. An application. We observe first that $h(z-\overline{\lambda}_n)^{-1}\in H^2$ in view of Lemma 1. Now we show how to deduce the main result of [**5**] from this theorem. Let $\mathcal{E}_a$ be the set of entire functions with conjugate indicator diagram $[0,ia]$, $\beta\in\mathbf{R}$, $\{\lambda_n\}$ a sequence satisfying $\inf_n \operatorname{Im}\lambda_n>0$, B the Blaschke product associated with $\{\lambda_n\}$, $\theta^a(z)=\exp(iaz)$.

THEOREM (GUBREEV [**5**]). *For $\beta\in(1/2,3/2)$ the family $\{x^{\beta-1}E_{1,\beta}(ix\lambda_n)\}$ is an unconditional basis in $L^2(0,a)$ if and only if*

(a) *$\{\lambda_n\}$ is an interpolating sequence; and*

(b) *there is an entire function F in $\mathcal{E}_a$ with simple zeros $\{\lambda_n\}$ such that $|x|^{2(\beta-1)}|F(x)|^2$ satisfies the Muckenhoupt condition.*

PROOF. This theorem is in fact a partial case of Theorem 1 with $h=z^{1-\beta}$. To see this we put $x_+\overset{\text{def}}{=}\max(x,0)$ and note that $(x_+^{\alpha-1}/\Gamma(\alpha))*(x_+^{\mu-1}/\Gamma(\mu)) = x_+^{\alpha+\mu-1}/\Gamma(\alpha+\mu)$ (see [**9**]). Hence, applying the operator of fractional integration

$$I_\alpha f(x)\overset{\text{def}}{=}\frac{x_+^{\alpha-1}}{\Gamma(\alpha)}*f=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)\,dt$$

with $\alpha = \beta - 1$ to the Taylor series of $\exp(i\lambda x)$, we obtain $I_{\rho-1}\exp(i\lambda x) = x^{\beta-1}E_{1,\beta}(ix\lambda)$. Next,

$$\frac{1}{\Gamma(\beta)}\int_0^\infty x^{\beta-1}e^{izx}\,dx = e^{i\beta\pi/2}z^{-\beta}, \qquad z \in \mathbf{C}_+$$

(see [**9**]), which yields the following formula for the Fourier-Laplace transform of $x^{\beta-1}E_{1,\beta}(ix\lambda)$, $\lambda \in \mathbf{C}_+$:

$$\int_0^\infty x^{\beta-1}E_{1,\beta}(ix\lambda)e^{izx}\,dx = e^{i\beta\pi/2}(\lambda+z)^{-1}\cdot z^{1-\beta}.$$

The Fourier-Laplace transform $f \to \hat{f}$ is an isometry of $L^2(0,+\infty)$ onto H^2 which maps $L^2(0,a)$ onto K_{θ^a}. The function $z^{1-\beta}$ is outer in $\mathbf{C}_+$ and $|x|^{2(1-\beta)}$ satisfies (1) for $\beta \in (\frac{1}{2},\frac{3}{2})$. Therefore we can put $h = z^{1-\beta}$ in Theorem 1.

To finish the proof we need the following lemma. For an outer function h satisfying $|h|^2 \in (1)$ we consider the class $\mathcal{M}_{h,a}$ of entire functions $F \in \mathcal{E}_a$ such that $|Fh^{-1}|^2 \in (1)$.

LEMMA 2. *Let $\{\lambda_n\}$ be a Blaschke sequence in $\mathbf{C}_+$ with $\inf \operatorname{Im}\lambda_n > 0$. Then the following are equivalent:*

(1) *There is an F in $\mathcal{M}_{h,a}$ with simple zeros $\{\lambda_n\}$.*

(2) *$\overline{h}\cdot h^{-1}\overline{B}\theta^a$ is a Helson-Szegö function.*

A proof of the lemma follows the same line as the proof of Theorem 1.2 (Part III of [**3**]), which deals with the case $h \equiv 1$. □

For $\beta \le \frac{1}{2}$ the family $\{x^{\beta-1}E_{1,\beta}(ix\lambda)\}$ is not a basis in $L^2(0,a)$ since no element of the family belongs to $L^2(0,a)$. To see that the case $\beta \ge \frac{3}{2}$ is also impossible is more difficult and we refer the interested reader to [**5**].

4. Proof of Theorem 1. Let N and M be linear subspaces of a linear space H. We say that H is a direct sum of N and M (notationally $H = N \dotplus M$) if $N \cap M = \{0\}$ and $H = N + M \overset{\text{def}}{=} \{n+m\colon n \in N, m \in M\}$.

We begin with two geometrical lemmas.

LEMMA 3. *Let N and M be closed subspaces of a Banach space H and T be a bounded operator from H to a Banach space X such that $\operatorname{Ker} T = N$ and $TH = X$. Then $H = N \dotplus M$ if and only if T maps M isomorphically onto X.*

PROOF. If $H = N \dotplus M$ then $T\colon M \to X$ is an isomorphism by the Banach theorem. Let now T map M isomorphically onto X. Pick any $h \in H$ and consider $m = (T|M)^{-1}Th \in M$. Then obviously $n = h - m \in \operatorname{Ker} T = N$, i.e. $h = n + m$. □

LEMMA 4. *The orthogonal projection P_G onto a subspace G of a Hilbert space H maps a subspace F isomorphically onto G if and only if $H = F \dotplus G^\perp$, where $G^\perp = H \ominus G$.*

PROOF. Put $X = G$, $T = P_G$ and apply Lemma 3. □

We are going to apply Lemma 3 with T being a Toeplitz operator. The following lemma justifies these applications.

LEMMA 5. *Let φ be a Helson-Szegö function and I an arbitrary inner function. Then $T_{\varphi\overline{I}}(H^2) = H^2$.*

PROOF. The desired equality is an immediate consequence of the factorization $T_{\varphi\overline{I}} = T_{\overline{I}}T_{\varphi}$. □

LEMMA 6. *Let h be an outer function and $\{\lambda_n\}$ an arbitrary Blaschke sequence in $\mathbf{C}_+$. Let $|h|^2 \in (1)$ and θ be an inner function. Then $\mathcal{P}_\theta$ is an isomorphism from the closed linear span of $\{h(z-\overline{\lambda}_n)^{-1}\}$ onto K_θ if and only if $\overline{h}\cdot h^{-1}\cdot\overline{B}\cdot\theta$ is a Helson-Szegö function, B being the Blaschke product with zeros $\{\lambda_n\}$.*

PROOF. Let F be the closed linear span of $\{h(z-\overline{\lambda}_n)^{-1}\}$ and $G = K_\theta$. By Lemma 4 the projection $\mathcal{P}_\theta$ maps F isomorphically onto G iff $H^2 = F \dotplus G^\perp$. Put now $N = F$, $M = G^\perp = \theta H^2$, $X = H = H^2$, $T = T_{(\overline{h}/h)\overline{B}}$ in Lemma 3. Lemma 5 guarantees that $T_{(\overline{h}/h)\overline{B}}(H^2) = H^2$. To check that $\operatorname{Ker} T_{(\overline{h}/h)\overline{B}} = F$ we observe that $T_{(\overline{h}/h)\overline{B}} = T_{\overline{B}}\cdot T_{\overline{h}/h}$. This yields $\operatorname{Ker} T_{(\overline{h}/h)\overline{B}} = (T_{(\overline{h}/h)})^{-1}(\operatorname{Ker} T_{\overline{B}})$. Now $\operatorname{Ker} T_{\overline{B}} = K_B = \operatorname{span}\{(z-\overline{\lambda}_n)^{-1}\}$. Besides, we have

$$T_{\overline{h}/h}(h(z-\overline{\lambda}_n)^{-1}) = \mathcal{P}_+(\overline{h}(z-\overline{\lambda}_n)^{-1}) = \overline{h(\lambda_n)}(z-\overline{\lambda}_n)^{-1}, \tag{4}$$

which implies the desired conclusion $\operatorname{Ker} T = F$, since $T_{\overline{h}/h}$ is invertible.

Lemma 3 says that $H^2 = F + G^\perp$ iff $T_{(\overline{h}/h)\overline{B}}\colon G^\perp \to H^2$ is an isomorphism. The latter is, clearly, equivalent to the fact that $(\overline{h}/h)\overline{B}\theta$ is a Helson-Szegö function. □

Let (a) and (b) of Theorem 1 hold. Then (a) implies that $\{(z-\overline{\lambda}_n)^{-1}\}$ is an unconditional basis in K_B [**10**] (see also [**3**]). Since $T_{\overline{h}/h}$ is invertible, it follows from (4) that $\{h(z-\overline{\lambda}_n)^{-1}\}$ is an unconditional basis in its span F. Now (b) and Lemma 6 imply that $\mathcal{P}_\theta$ maps F isomorphically onto K_θ, which proves the theorem in one direction.

Let $\{\mathcal{P}_\theta(h(z-\overline{\lambda}_n)^{-1})\}$ be an unconditional basis in K_θ. We recall that a family $\{e_n\}$ of vectors of a Banach space is called uniformly minimal if there is a biorthogonal family $\{f_n\}$ of bounded functionals ($\langle e_n, f_m\rangle = \delta_{nm}$) such that $\sup_n \|e_n\|\cdot\|f_n\| < +\infty$. If $\{e_n\}$ is an unconditional basis then it is uniformly minimal. It is clear that if $Te_n^* = e_n$ and $\|e_n\| \ge c\cdot\|e_n^*\|$ then $\{e_n^*\}$ is also uniformly minimal. Set $f_n^* = T^*f_n$. Then $\|e_n^*\|\cdot\|f_n^*\| \le c^{-1}\cdot\|T^*\|\cdot\|e_n\|\cdot\|f_n\|$.

LEMMA 7. *If $\sup_n|\theta(\lambda_n)| < 1$ then there is a positive constant c such that*

$$\|\mathcal{P}_\theta(h(z-\overline{\lambda}_n)^{-1})\| \ge c\|h(z-\overline{\lambda}_n)^{-1}\|.$$

PROOF. We have

$$\begin{aligned}\|\mathcal{P}_\theta(h(z-\overline{\lambda}_n)^{-1}\| &= \|(I-\mathbf{P}_+)(h\overline{\theta}(x-\overline{\lambda}_n)^{-1}\|_{L^2} = \|\mathbf{P}_+\theta\overline{h}(x-\lambda_n)^{-1}\|_{L^2},\\ &= \left\|\mathbf{P}_+\frac{\overline{h}}{h}\cdot\frac{\theta(x)-\theta(\lambda_n)}{x-\lambda_n}\cdot h\right\| \ge (\|T_{\overline{h}/h}^{-1}\|)^{-1}\cdot\left\|\frac{\theta(x)-\theta(\lambda_n)}{x-\lambda_n}h\right\|_{L^2}\\ &\ge \left(1-\sup_n|\theta(\lambda_n)|\right)\cdot(\|T_{\overline{h}/h}^{-1}\|)^{-1}\cdot\|h(x-\overline{\lambda}_n)^{-1}\|_{H^2}.\end{aligned}$$

Since $T_{\overline{h}/h}$ is invertible, (4) implies that $\{(z-\overline{\lambda}_n)^{-1}\}$ is uniformly minimal too. It is well known that $\{B(z-\lambda_n)^{-1}(iB'(\lambda_n))^{-1}\}$ is a biorthogonal family for

$\{(z-\overline{\lambda}_n)^{-1}\}$, which yields $\inf_n |B'(\lambda_n)|/\operatorname{Im}\lambda_n > 0$ and therefore $\{\lambda_n\}$ is interpolating. Then $\{(z-\overline{\lambda}_n)^{-1}\}$ is an unconditional basis in K_B. From (4) we obtain that $\{h(z-\overline{\lambda}_n)^{-1}\}$ is an unconditional basis in F because $T_{\overline{h}/h}$ is invertible. Lemma 7 says that $\mathcal{P}_\theta$ does not distort the elements $h(z-\overline{\lambda}_n)^{-1}$. This clearly implies that $\mathcal{P}_\theta\colon F \to K_\theta$ is an isomorphism. Lemma 6 gurantees that $\overline{h}\cdot h^{-1}\cdot \overline{B}\theta$ is a Helson-Szegö function. □

References

1. B. S. Pavlov, *Basicity of an exponential system and Muckenhoupt condition*, Soviet Math. Dokl. **20** (1979), 655–659.
2. N. K. Nikol'skiĭ, *Bases of exponential functions and values of reproducing kernels*, Soviet Math. Dokl. **21** (1980), 937–941.
3. S. V. Hruščëv, N. K. Nikol'skiĭ, and B. S. Pavlov, *Unconditional bases of exponentials and of reproducing kernels*, Lecture Notes in Math., vol. 864, Springer-Verlag, Berlin and New York, 1981, pp. 214–335.
4. S. V. Hruščëv, *The Regge problem for strings, unconditionally convergent eigenfunction expansions, and unconditional bases of exponentials in $L^2(-T,T)$*, J. Operator Theory **14** (1985), 67–85.
5. G. M. Gubreev, *A basis property of families of the Mittag-Leffler functions, Dzhrbashyan's transform, and the Muckenhoupt condition*, Funktsional. Anal. i Prilozhen. **20**:3 (1986).
6. A. Erdélyi, *Higher transcendental functions*, vol. 3 (Bateman Manuscript Project), McGraw-Hill, New York.
7. H. Widom, *Inversion of Toeplitz matrices*. III, Notices Amer. Math. Soc. **7** (1960), 63.
8. J. B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
9. I. M. Gel'fand and G. E. Shilov, *Generalized functions*, vol. 1, Academic Press, New York, 1964.
10. N. K. Nikol'skiĭ and B. S. Pavlov, *Eigenvector bases, characteristic functions and interpolation problems in Hardy's H^2-space*, Soviet Math. Dokl. **10** (1969), 138–141.

Leningrad Branch of the V. A. Steklov Mathematical Institute, Fontanka 27, 191011 Leningrad D-11, USSR