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NOTE ON COMPATIBLE VECTOR TOPOLOGIES

JERZY KĄKOL

ABSTRACT. Let $\langle X, Y\rangle$ be a dual pair. Then X admits the finest locally convex topology μ which is compatible with $\langle X, Y\rangle$. In contrast, it is proved that there is no finest vector topology on X which is compatible with $\langle X, Y\rangle$ provided X contains a μ-dense subspace of infinite codimension.

Introduction. Let τ be a vector topology on a (vector) space X different from the finest one of X and compatible with a dual pair $\langle X, Y\rangle$, i.e. Y is the topological dual of (X, τ).

(a) *Does there exist on X a vector topology which is strictly finer than τ and compatible with $\langle X, Y\rangle$?*

(b) *Does there exist on X the finest vector topology compatible with $\langle X, Y\rangle$?*

We prove that (a) has a positive and (b) a negative solution whenever X contains a τ-dense subspace of infinite codimension. In fact we obtain a stronger result (Theorem 2). Some applications are also included.

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Notation. We consider only infinite dimensional Hausdorff topological vector spaces (tvs) $X = (X, \tau)$. If G is a (vector) subspace of X, then $\tau \mid G$ and τ/G denote the topology τ restricted to G and the quotient topology of the quotient space X/G, respectively. If λ is a finer vector topology on X/G, we denote by $\vartheta := \tau \vee \lambda$ the weakest vector topology on X such that $\tau \le \vartheta$, $\vartheta/G = \lambda$, $\vartheta \mid G = \tau \mid G$ (cf. [**2**]). By $\sup(\tau, \vartheta)$ [$\inf(\tau, \vartheta)$] we denote the weakest [finest] vector topology on X which is finer [weaker] than τ and ϑ. A tvs X (and its topology) will be called *dual-less* if $X' = 0$; X' and X^* denote the topological and algebraic dual of X, respectively. A tvs X is *semibornological* [*Mazur*] if every bounded [sequentially continuous] linear functional on X is continuous.

Results. We shall need the next lemma; its proof combines some ideas found in [**7** and **5**].

LEMMA 1. *Every infinite dimensional vector space X admits a locally bounded dual-less topology.*

PROOF. Let Γ be a Hamel basis of X. Let ϑ be a vector topology defined by the norm $\|x\| := \sum |t_s|$, where $x = \sum t_s x_s$, $x_s \in \Gamma$. Clearly (X, ϑ) is isomorphic to a dense subspace of the space $l^1(\Gamma)$. Hence it is enough to find on $l^1(\Gamma)$ a weaker

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locally bounded dual-less topology. Fix $0 < p < 1$. Choose in the dual-less space $L^p := (L^p[0,1], \| \|_p)$ a sequence (y_n) with the properties: $\sum \|y_n\|_p < \infty$; $\mathrm{lin}(y_n)$ is dense in L^p; and (y_n) is m-independent, i.e. if $(t_n) \in l^\infty$ and $\sum t_n y_n = 0$, then $(t_n) = 0$ [**5**, Theorem 1]. If $T(x) := \sum x_n y_n$, $x = (x_n) \in l^1$, then T is a continuous injective linear map with dense range. T induces a continuous injective linear map $\hat{T}\colon l^1(\Gamma, l^1) \to l^1(\Gamma, L^p)$ also with dense range, where $l^1(\Gamma, L^p)$ is the locally bounded dual-less space of all functions $f = (f_s)$, $f_s \in L^p$, with $\sum \|f_s\|_p < \infty$. Since $l^1(\Gamma)$ and $l^1(\Gamma, l^1)$ are isomorphic, $l^1(\Gamma)$ admits a topology as claimed.

THEOREM 2. *Let G be a dense infinite codimensional subspace of a tvs (X, τ). Then X admits a strictly finer vector topology ϑ such that $\tau \mid G = \vartheta \mid G$, ϑ/G is locally bounded and dual-less, and $(X,\tau)' = (X,\vartheta)'$. Moreover, there is no finest vector topology α on X such that $(X,\tau)' = (X,\alpha)'$.*

PROOF. Set $\vartheta := \tau \vee \varphi$, where φ is a vector topology on X/G as in Lemma 1. To finish the proof it is enough to find on X strictly finer vector topologies producing the same continuous linear functionals as τ, but whose supremum topology does not have this property. In view of Theorem B of [**7**] the finest vector topology γ on X/G is the supremum of dual-less topologies $\gamma^1, \gamma^2, \gamma^3$. Set $\vartheta^i := \tau \vee \gamma^i$ and $\vartheta := \sup(\vartheta^i\colon 1 \le i \le 3)$. Clearly $(X, \vartheta^i)' = (X,\tau)'$, $i = 1,2,3$. Moreover, $\vartheta/G = \gamma$. Then $(X,\tau)' \neq (X,\vartheta)'$ (because every nontrivial linear functional on X which vanishes on G is ϑ-continuous but discontinuous for τ).

Note that there exist tvs X not carrying the finest vector topology of X but which admit the finest vector topology compatible with $\langle X, X'\rangle$; every uncountable dimensional vector space X with the weak topology $\sigma(X, X^*)$ provides an example of such a space.

REMARK 3. Note that a tvs (X,τ) has a dense subspace of infinite codimension if X contains an infinite dimensional subspace admitting a finer metrizable vector topology, compare [**6**, Theorem 1]. In particular, every boundedly summing tvs [**1**, p. 74] containing an infinite dimensional bounded subset has a dense subspace of infinite codimension (this generalizes Proposition 1.5 of [**9**]). Note that every bornological (or boundedly summing ultrabornological [**4**]) space X with $X' \neq X^*$ enjoys this property, compare [**1**, (5), p. 76]. In particular, every semibornological lcs X with $X' \neq X^*$ also has this property (because X is bornological under the finest locally convex topology τ^b which produces the same bounded sets as τ; clearly then $(X,\tau)' = (X,\tau^b)'$).

A vector topology τ on X is said to have the *Hahn Banach Extension Property* (HBEP) if $(X,\tau)'$ separates points from the closed subspaces of X.

PROPOSITION 4. *Let X be an lcs such that (X,μ) contains a dense bornological subspace of infinite codimension. Then there exist on X two vector topologies ϑ^1 and ϑ^2 without the HBEP, compatible with $\langle X, X'\rangle$, such that $\mu = \inf(\vartheta^1, \vartheta^2)$. Moreover, ϑ^1 and ϑ^2 can be chosen to be metrizable [and ultrabarrelled] provided μ is metrizable [and complete].*

PROOF. Let ϑ^1 be a vector topology on X as in Theorem 2. Using Remark 3 we deduce that (X,ϑ^1) has a dense subspace G of infinite codimension. By Theorem 2 we find on X a vector topology ϑ^2 which is compatible with $\langle X, X'\rangle$ and not compatible with ϑ^1. Set $\gamma := \inf(\vartheta^1, \vartheta^2)$. Since G is γ-dense, γ is locally convex. Hence $\mu = \gamma$. Applying Theorem 2.6 of [**8**] we obtain the last assertion of Proposition 4.

Applying Corollary 1.3 of [**10**] and our Remark 3 we deduce that every sequentially complete bornological space X with $X' \neq X^*$ satisfies also the assumption of Proposition 4.

By the *three space problem* (for property P) we understand the following: Suppose X is a tvs and G is a subspace of X such that G and X/G have property P. Does X have property P? (see for example [**8**]).

The following fact (being an immediate consequence of the proof of Theorem 2) will be used to establish that the three space problem has a negative solution if P is either semibornological or Mazur.

PROPOSITION 5. *Let* P *and* P^0 *be certain properties of tvs such that:* (1) *Every metrizable tvs has property* P. (2) *If* (X, τ) *has property* P, *then* (X, τ^0) *has property* P^0, *where* τ^0 *denotes the finest locally convex topology on* X *weaker than* τ. *If there exists a Mackey space* (X, τ), *i.e.* $\tau = \mu$, *without property* P^0 *but containing a dense infinite codimensional subspace with property* P, *the three space problem has a negative solution for property* P.

COROLLARY 6. *The three space problem has a negative solution for* P *semibornological* [*Mazur*].

PROOF. A slight modification of Example 3 of [**3**] provides an example of a barrelled (not Mazur) space X containing a dense ultrabornological (hence semibornological and Mazur) subspace G with $\dim(X/G) = \aleph_0$.

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INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY, MATEJKI 48/49, 60-769 POZNAŃ, POLAND