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ASYMPTOTICALLY PERIODIC SOLUTIONS OF A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we give necessary and sufficient conditions for all solutions of the system

$$\text{(S)} \qquad x' = y, \qquad y' = -a(t)f(x)g(y)$$

to be oscillatory or bounded, for all orbits of the system

$$(\mathrm{S}_1) \qquad x' = y, \qquad y' = -\alpha f(x)g(y)$$

to be periodic, where $a(t) \to \alpha > 0$ as $t \to \infty$, and for every orbit of (S) to approach a periodic orbit of (S_1). The conditions assuring that every solution of (S) is asymptotically periodic are also established.

1. Introduction. We consider the second order differential equation

$$\text{(E)} \qquad x'' + a(t)f(x)g(x') = 0,$$

where $a\colon I \to R^+ = (0,\infty)$, $I = [\tau,\infty)$, $f\colon R \to R = (-\infty,\infty)$ and $g\colon R \to R^+$ are continuous functions, $a(t) \to \alpha > 0$ as $t \to \infty$, and $xf(x) > 0$ for $x \neq 0$. Assume that the solution of any Cauchy problem is unique and can be defined on I.

We also consider the limit equation of (E),

$$(\mathrm{E}_1) \qquad x'' + \alpha f(x)g(x') = 0,$$

and the equivalent systems of (E) and (E_1),

$$\text{(S)} \qquad x' = y, \qquad y' = -a(t)f(x)g(y)$$

and

$$(\mathrm{S}_1) \qquad x' = y, \qquad y' = -\alpha f(x)g(y).$$

The purpose of the present paper is to establish necessary and sufficient conditions for all solutions of (S) to be oscillatory or bounded, for all orbits of (S_1) to be periodic, and for every orbit of (S) to approach a periodic orbit of (S_1) as $t \to \infty$, and also conditions to insure that every solution of (S) is asymptotically periodic. Our results improve and extend some theorems in [**1–8**].

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2. Equation (E).

THEOREM 1. (E) *is oscillatory if and only if*

$$\mathrm{H}_1:\quad F(x)=\int_0^x f(u)\,du\to\infty \quad \textit{as } |x|\to\infty,$$

$$\mathrm{H}_2:\quad G(y)=\int_0^y (v/g(v))\,dv\to\infty \quad \textit{as } |y|\to\infty.$$

PROOF. Sufficiency follows directly from [**3**, Theorem 1].

Necessity. If $F(\infty)<\infty$, then for given $x_0'>0$, there exists $x_0>0$ such that

$$F(\infty)-F(x_0)<\frac{1}{\gamma}G(x_0'), \tag{1}$$

where $\gamma=\sup_{t\geq\tau}a(t)$. By assumption on $a(t)$ clearly $0<\gamma<\infty$.

Let $x(t)$ be a solution of Cauchy problem (E) with $x(\tau)=x_0$, $x'(\tau)=x_0'$. By assumption, $x(t)$ and its derivative $x'(t)$ are oscillatory. Let T_1 be the first zero of $x'(t)$ on I. Clearly, $\tau<T_1<\infty$ and $x(t)>0$ for $t\in[\tau,T_1]$.

Integrating (E) from τ to T_1, we get

$$\begin{aligned} G(x'(T_1))-G(x'(\tau)) &= -\int_\tau^{T_1} a(t)f(x(t))x'(t)\,dt \\ &\geq -\gamma[F(x(T_1))-F(x(\tau))]. \end{aligned}$$

Then

$$G(x_0')\leq\gamma(F(\infty)-F(x_0)),$$

which contradicts (1). For the case $F(-\infty)<\infty$, the proof is similar, so H_1 is necessary.

If $G(\infty)<\infty$, then for given $x_0<0$, there exists $x_0'>0$ such that

$$G(\infty)-G(x_0')<\beta F(x_0), \tag{2}$$

where $\beta=\inf_{t\geq\tau}a(t)$. Clearly $0<\beta<\infty$.

Let $x(t)$ be a solution of Cauchy problem (E) with $x(\tau)=x_0$, $x'(\tau)=x_0'$. By assumption, $x(t)$ and $x'(t)$ are oscillatory. Let T_2 be the first zero of $x(t)$ on I. Clearly $\tau<T_2<\infty$ and $x'(t)>0$ for $t\in[\tau,T_2]$.

Integrating (E) from τ to T_2, we get

$$\begin{aligned} G(x'(T_2))-G(x'(\tau)) &= -\int_\tau^{T_2} a(t)f(x(t))x'(t)\,dt \\ &\geq -\beta[F(x(T_2))-F(x(\tau))]. \end{aligned}$$

Then

$$G(\infty)-G(x_0')\geq\beta F(x_0),$$

which contradicts (2). For the case $G(-\infty)<\infty$, the proof is similar, so H_2 is necessary. This completes the proof.

REMARK 1. The sufficient condition of Theorem 1 extends [**1**, Theorem 0.1 and **5**, Theorem 2].

Let $x(t)$ be a nontrivial oscillatory solution of (E). Clearly, $x(t)$ and $x'(t)=y(t)$ are oscillatory and their zeros separate one another. Let $\{t_{2n}\}$ and $\{t_{2n+1}\}$ be sequences of zeros of $x(t)$ and $y(t)$ respectively such that $t_{2n} < t_{2n+1} < t_{2n+2}$ $(n = 0, 1, \ldots)$, $t_n \to \infty$ as $n \to \infty$.

Integrating (S) from t_{2k-1} to t_{2k}, we obtain

$$G(y(t_{2k})) = a(\tau_{2k-1})F(x(t_{2k-1})) \qquad (t_{2k-1} < \tau_{2k-1} < t_{2k},\ k = 1, 2, \ldots). \tag{3}$$

Similarly, integrating (S) from t_{2k} to t_{2k+1}, we obtain

$$G(y(t_{2k})) = a(\tau_{2k})F(x(t_{2k+1})) \qquad (t_{2k} < \tau_{2k} < t_{2k+1},\ k = 0, 1, \ldots). \tag{4}$$

Denote

$$s_n = \prod_{k=1}^{n} \frac{a(\tau_{2k-1})}{a(\tau_{2k})}, \quad s'_n = \prod_{k=1}^{n} \frac{a(\tau_{2k-1})}{a(\tau_{2k-2})} \qquad (n = 1, 2, \ldots).$$

The sequences $\{s_n\}$ and $\{s'_n\}$ are called characteristic sequences of the oscillatory solution $x(t)$. Clearly,

$$s'_n = \frac{a(\tau_{2n})}{a(\tau_0)} s_n \qquad (n = 1, 2, \ldots). \tag{5}$$

We have the following lemma.

LEMMA 1. *Suppose that $x(t)$ is a nontrivial oscillatory solution of* (E). *Then the sequences $\{x(t_{2n+1})\}$ and $\{x'(t_{2n})\}$ satisfy* (3) *and* (4), *the sequences $\{s_n\}$ and $\{s'_n\}$ satisfy* (5).

LEMMA 2. *Suppose that all solutions of* (E) *are bounded. Then* (E) *is oscillatory.*

Lemma 2 is an immediate corollary of [**4**, Theorem 1].

We need the following hypothesis:

H_3: (E) is oscillatory and the characteristic sequence $\{s_n\}$ of every nontrivial solution of (E) is bounded.

THEOREM 2. *All solutions of* (E) *and their derivatives are bounded if and only if H_1–H_3 hold.*

PROOF. *Sufficiency.* By H_1, H_2, and Theorem 1, we know that any solution $x(t)$ of (E) and $x'(t)$ are oscillatory and hence by Lemma 1, (3)–(5) hold. From (3) and (4) we obtain

$$F(x(t_{2n+1})) = s_n F(x(t_1)) \qquad (n = 1, 2, \ldots) \tag{6}$$

and

$$G(x'(t_{2n})) = s'_n G(x'(t_0)) \qquad (n = 1, 2, \ldots). \tag{7}$$

From (6) and boundedness of the sequence $\{s_n\}$, we get

$$F(x(t_{2n+1})) \le sF(x(t_1)) \qquad (n = 1, 2, \ldots),$$

where $s_n \le s < \infty$. From this estimate and H_1 it follows that the sequence $\{x(t_{2n+1})\}$ is bounded, so that $x(t)$ is bounded.

From (5) and H_3, we have

$$s'_n \le \gamma s / a(\tau_0) \qquad (n = 1, 2, \ldots),$$

where $\gamma = \sup_{t\geq\tau} a(t) > 0$, and in view of (7)

$$G(x'(t_{2n})) \leq \frac{\gamma s}{a(\tau_0)} G(x'(t_0)) \qquad (n = 1, 2, \ldots).$$

From this estimate and H_2 it follows that the sequence $\{x'(t_{2n})\}$ is bounded, so that $x'(t)$ is bounded.

Necessity. By Lemma 2 (E) is oscillatory, and hence by Theorem 1 H_1 and H_2 hold. Let $x(t)$ be any bounded oscillatory solution of (E). From (6) we get

$$s_n = F(x(t_{2n+1}))/F(x(t_1)) \leq L/F(x(t_1)) \qquad (n = 1, 2, \ldots),$$

where $L = \max_{|x|\leq M} F(x)$ and $M = \sup_{t\geq\tau} |x(t)|$. Thus $\{s_n\}$ is bounded. This completes the proof.

COROLLARY 1. *Suppose that the following holds.*

H_4: *$a(t)$ is a function of bounded variation on I.*

Then all solutions of (E) *and their derivatives are bounded if and only if* H_1 *and* H_2 *hold.*

In fact, under the assumption on $a(t)$, it can be shown that every characteristic sequence $\{s_n\}$ is bounded, for which it is sufficient to show the convergence of the infinite product

$$s_\infty = \prod_{k=1}^{\infty} \frac{a(\tau_{2k-1})}{a(\tau_{2k})},$$

which is equivalent to the convergence of the series

$$\sigma = \sum_{k=1}^{\infty} \frac{a(\tau_{2k-1}) - a(\tau_{2k})}{a(\tau_{2k})}.$$

By $a(t) \to \alpha$ as $t \to \infty$ and H_4, we have that

$$\sum_{k=1}^{\infty} \left| \frac{a(\tau_{2k-1}) - a(\tau_{2k})}{a(\tau_{2k})} \right| = \frac{1}{\beta} \sum_{k=1}^{\infty} |a(\tau_{2k-1}) - a(\tau_{2k})| < \infty,$$

where $\beta = \inf_{t\geq\tau} a(t) > 0$. This implies that the series σ is absolutely convergent and hence s_∞ is convergent. So $\{s_n\}$ is bounded.

REMARK 2. The sufficiency condition of Corollary 1 improves and extends [**2**, Theorems 8 and 9; **7**, Theorems 4 and 5; and **8**, Theorems 1–3 and their corollary].

3. Equation (E_1). We note that there is one-to-one correspondence between periodic solutions of (E_1) and periodic orbits of (S_1), and that every orbit of (S_1) is round the origin.

THEOREM 3. *Every solution of* (E_1) *is periodic if and only if* H_1 *and* H_2 *hold.*

PROOF. *Necessity.* Since every periodic solution of (E_1) is oscillatory, by Theorem 1 we have H_1 and H_2.

Sufficiency. Consider the function $V(x,y) = \alpha F(x) + G(y)$. Taking the derivative of $V(x,y)$ along an orbit of (S_1) we get

$$V'(x,y) = \alpha f(x) y + (y/g(y))(-\alpha f(x) g(y)) \equiv 0.$$

Hence, from H_1 and H_2 it follows that $V(x,y) = C$ is a closed orbit of (S_1) for each $C > 0$. This completes the proof.

REMARK 3. If $g(y) \equiv 1$, then [**1**, Theorem 0.1 and **5**, Theorem 2] can be derived from the sufficiency of Theorem 3.

4. Equations (E) and (E$_1$). In this and the next section we need the following hypothesis:

H$_5$: (E) is oscillatory and the characteristic sequence $\{s_n\}$ of every nontrivial solution of (E) satisfies $s_n \to s > 0$ as $n \to \infty$.

THEOREM 4. *Every orbit of* (S) *approaches a periodic orbit of* (S$_1$) *in a spiral manner as* $t \to \infty$ *if and only if* H$_1$, H$_2$, *and* H$_5$ *hold.*

PROOF. *Sufficiency.* By H$_1$, H$_2$, and Theorem 3 every orbit of (S$_1$) is closed. By Theorems 1 and 2 we know that every solution $x(t)$ of (E) and $x'(t) = y(t)$ are oscillatory and bounded, and hence (6), (7), and (5) hold. Letting $n \to \infty$ in (6) and (7), by H$_5$ and (5) we have

$$\lim_{n\to\infty} F(x(t_{2n+1})) = sF(x(t_1)) \tag{8}$$

and

$$\lim_{n\to\infty} G(y(t_{2n})) = s'G(y(t_0)), \tag{9}$$

where $s' = \alpha s/a(\tau_0) > 0$. Letting $k \to \infty$ in (4), by (8) and (9) we find

$$\alpha sF(x(t_1)) = s'G(y(t_0)) = C_1 > 0.$$

Consider the periodic orbit of (S$_1$):

$$V(x,y) = \alpha F(x) + G(y) = C_1. \tag{10}$$

Clearly, $V(x,y) = C_1$ intersects the x-axis at $(x_1, 0)$ and $(x_2, 0)$, and the y-axis at $(0, y_1)$ and $(0, y_2)$. Without loss of generality we assume that $x_1 > 0$, $x_2 < 0$, $y_1 > 0$, and $y_2 < 0$, and $x(t_{4n-3}) > 0$, $x(t_{4n-1}) < 0$, $y(t_{4n-2}) > 0$, and $y(t_{4n}) < 0$. From (8)–(10) it follows that $x(t_{4n-3}) \to x_1$, $x(t_{4n-1}) \to x_2$, $y(t_{4n-2}) \to y_1$, and $y(t_{4n}) \to y_2$ as $n \to \infty$. We now show that for arbitrary $\varepsilon > 0$, there exists $\tau' \geq \tau$ such that

$$M(t) = (x(t), y(t)) \in A(\varepsilon) = \{(x,y)\colon C_1 - \varepsilon < V(x,y) < C_1 + \varepsilon\}$$

for all $t \geq \tau'$. In fact, assume the contrary. Then by boundedness of $M(t)$, there exists a sequence $\{t'_j\}$, $t'_j \to \infty$ as $j \to \infty$, such that $M(t'_j) \notin A(\varepsilon)$ and $M(t'_j) \to \overline{M} = (\overline{x}, \overline{y})$ as $j \to \infty$. Clearly, $\overline{M} \notin A(\varepsilon)$. Without loss of generality we assume $\overline{x} > 0$, $\overline{y} > 0$. Then $t_{4n_j-3} < t'_j < t_{4n_j-2}$ for j sufficiently large. Integrating (S) from t_{4n_j-3} to t'_j we obtain

$$G(y(t'_j)) + a(\tau'_{4n_j-3})F(x(t'_j)) = a(\tau'_{4n_j-3})F(x(t_{4n_j-3})) \qquad (t_{4n_j-3} < \tau'_{4n_j-3} < t'_j). \tag{11}$$

Letting $j \to \infty$ in (11), we have $V(\overline{x}, \overline{y}) = C_1$, which is impossible.

Necessity. By assumption we know that any solution $(x(t), y(t))$ of (S) is oscillatory and bounded, and hence by Theorem 2 H$_1$ and H$_2$ hold.

As before, we assume $x(t_{4n-3}) \to x_1 > 0$ and $x(t_{4n-1}) \to x_2 < 0$ as $n \to \infty$, and that $V(x_1, 0) = V(x_2, 0) = C_1 > 0$, from which we have $F(x_1) = F(x_2) = C_1/\alpha > 0$. On the other hand, from (6) it follows that

$$s_{2k-2} = F(x(t_{4k-3}))/F(x(t_1)), \tag{12}$$

and

$$s_{2k-1} = F(x(t_{4k-1}))/F(x(t_1)). \tag{13}$$

Letting $k \to \infty$ in (12) and (13), we obtain

$$\lim_{k\to\infty} s_{2k-2} = F(x_1)/F(x(t_1)) = C_1/[\alpha F(x(t_1))]$$

and

$$\lim_{k\to\infty} s_{2k-1} = F(x_2)/F(x(t_1)) = C_1/[\alpha F(x(t_1))].$$

Then we have

$$\lim_{n\to\infty} s_n = C_1/[\alpha F(x(t_1))] > 0.$$

This completes the proof.

COROLLARY 2. *Suppose that* H_4 *holds. Then every orbit of* (S) *approaches a periodic orbit of* (S_1) *in a spiral manner as* $t \to \infty$ *if and only if* H_1 *and* H_2 *hold.*

In fact, as in the proof of Corollary 1, the infinite product s_∞ is convergent. From this and $a(t) > 0$ for $t \in I$ it follows that $s_\infty > 0$. That is equivalent to $s_n \to s_\infty$ as $n \to \infty$ so that Corollary 2 can be derived from Theorem 4.

REMARK 4. The sufficiency condition of Corollary 2 improves [**6**, Theorems 5 and 6].

5. Asymptotically periodic solutions. A solution $(x(t), y(t))$ of (S) is defined to be asymptotically periodic if there exists a constant $T > 0$ such that for arbitrary $\varepsilon > 0$, there is a $\tau' \geq \tau$ so that

$$|(x(t+T), y(t+T)) - (x(t), y(t))| < \varepsilon \quad \text{for all } t \geq \tau'$$

(see [**6**, §5]).

THEOREM 5. *Suppose that* H_1, H_2, *and* H_5 *hold. Then every solution* $(x(t), y(t))$ *of* (S) *is asymptotically periodic.*

PROOF. As in the proof of Theorem 4, we assume that $x(t_{4n-3}) \to x_1 > 0$, $x(t_{4n-1}) \to x_2 < 0$, $y(t_{4n-2}) \to y_1 > 0$, and $y(t_{4n}) \to y_2 < 0$ as $n \to \infty$. Clearly, $(x(t + t_{4n-3}), y(t + t_{4n-3}))$ is a solution of the Cauchy problem

$$u' = v, \;\; v' = -\alpha f(u)g(v) + (\alpha - a(t + t_{4n-3}))f(x(t + t_{4n-3}))g(y(t + t_{4n-3}))$$

with $u(0) = x(t_{4n-3})$, $v(0) = 0$. Let $(x^*(t), y^*(t))$ be a solution of the Cauchy problem (S_1) with $x^*(0) = x_1$, $y^*(0) = 0$. By Theorem 3, $(x^*(t), y^*(t))$ is a periodic solution of (S_1) (periodic $T > 0$). Using Yoshizawa's method (see [**9**, §13]), it is easy to prove that $(x(t + t_{4n-3}), y(t + t_{4n-3}))$ converges to $(x^*(t), y^*(t))$ uniformly in $t \in [0, 3T]$ as $n \to \infty$. Hence, for $\varepsilon > 0$, there exists an integer $N > 0$ such that $t_{4(n+1)-3} - t_{4n-3} < 2T$ and $|(x(t + t_{4n-3}), y(t + t_{4n-3})) - (x^*(t), y^*(t))| < \varepsilon/2$ for $t \in [0, 3T]$ and $n \geq N$. From this we have

$$\begin{aligned}
&|(x(t + T + t_{4n-3}), y(t + T + t_{4n-3})) - (x(t + t_{4n-3}), y(t + t_{4n-3}))| \\
&\quad \leq |(x(t + T + t_{4n-3}), y(t + T + t_{4n-3})) - (x^*(t + T), y^*(t + T))| \\
&\qquad + |(x^*(t), y^*(t)) - (x(t + t_{4n-3}), y(t + t_{4n-3}))| \\
&\quad < \varepsilon \quad \text{for } t \in [0, 2T].
\end{aligned}$$

Furthermore, we obtain

$$|(x(t+T), y(t+T)) - (x(t), y(t))| < \varepsilon \quad \text{for all } t \geq t_{4N-3}.$$

This completes the proof.

COROLLARY 3. *Suppose that* H_1, H_2, *and* H_4 *hold. Then the conclusion of Theorem* 5 *holds.*

REMARK 5. Corollary 3 improves [**6**, Theorem 7 and its Corollary].

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