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LINEAR SUMS OF CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. Let f belong to a certain subclass of the class of functions which are regular in the unit disc $E = \{z\colon |z| < 1\}$. Suppose that $\phi = \phi(f, f', f'')$ and $\psi = \psi(f, f', f'')$ are regular in E with $\operatorname{Re}\phi > 0$ in E and $\operatorname{Re}\psi \not> 0$ in the whole of E. In this paper we consider the following two new types of problems: (i) To find the ranges of the real numbers λ and μ such that $\operatorname{Re}(\lambda\phi + \mu\psi) > 0$ in E. (ii) To determine the largest number $\rho, 0 < \rho < 1$, such that $\operatorname{Re}(\phi + \psi) > 0$ in $|z| < \rho$.

1. Introduction. Let A denote the class of functions f that are regular in the unit disc $E = \{z\colon |z| < 1\}$ and are normalized by the conditions $f(0) = f'(0) - 1 = 0$. We shall denote by S the subclass of A whose members are univalent in E. A function f belonging to S is said to be starlike of order α, $0 \le \alpha < 1$, if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$, $z \in E$, and we denote by $S_t(\alpha)$ the class of all such functions. $S_t = S_t(0)$ will be referred to as the class of starlike functions. Finally, we shall denote by K the class of convex functions, consisting of those elements $f \in S$ which satisfy the condition $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ in E. It is well known that $K \subset S_t(1/2)$.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are regular in E, then their Hadamard product/convolution is the function denoted by $f * g$ and defined by the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

It is known that $f * g$ is also regular in E.

Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be regular in E. Then the de la Valleé Poussin mean of f of order n, $V_n(z, f)$, is defined by

$$V_n(z,f) = \frac{n}{n+1} a_1 z + \frac{n(n-1)}{(n+1)(n+2)} a_2 z^2 + \cdots + \frac{n(n-1)(n-2)\cdots 2\cdot 1}{(n+1)(n+2)\cdots(2n)} a_n z^n.$$

Let f be regular in E and g regular and univalent in E with $f(0) = g(0)$. We say that f is subordinate to g in E (in symbols $f \prec g$ in E) if $f(E) \subset g(E)$.

A sequence $\{c_n\}_1^\infty$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is regular and convex in E, we have

$$\sum_{n=1}^{\infty} c_n a_n z^n \prec f(z) \quad \text{in } E.$$

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In the present paper we shall mainly be concerned with the following two new types of problems:

(a) If $\phi = \phi(f, f', f'')$ and $\psi = \psi(f, f', f'')$, where $f \in K$ or $S_t(1/2)$, such that $\operatorname{Re}\phi > 0$ in E and $\operatorname{Re}\psi$ is not necessarily positive in the whole of the unit disc E, to find the ranges of real numbers λ and μ such that $\operatorname{Re}(\lambda\phi + \mu\psi) > 0$ in E.

(b) To find the largest number ρ, $0 < \rho < 1$, such that $\operatorname{Re}(\phi + \psi) > 0$ in $|z| < \rho$.

2. Preliminary results. We shall need the following results, which we state as lemmas.

LEMMA 1. *If $f \in K$ and $g \in S_t$, then $(f * gF)/(f * g)$ takes values in the convex hull of $F(E)$ for every function F regular in E.*

LEMMA 2. *If f and g belong to $S_t(1/2)$, then $(f * gF)/(f * g)$ takes values in the convex hull of $F(E)$ for every function F regular in E.*

LEMMA 3. *A sequence $\{c_n\}_1^\infty$ of complex numbers is a subordinating factor sequence if and only if* $\operatorname{Re}(1 + 2\sum_{n=1}^{\infty} c_n z^n) > 0$ $(|z| < 1)$.

Lemmas 1 and 2 are due to Ruscheweyh and Sheil-Small [**1**] and Lemma 3 is due to Wilf [**2**].

3. Theorems and their proofs. It is well known [**1**] that if $f \in S_t(1/2)$, then $\operatorname{Re}(f(z)/s_n(z, f)) > 1/2$, $z \in E$, where $s_n(z, f)$ denotes the nth partial sum of f. From this it follows that given $f \in S_t(1/2)$ and any two real numbers $\lambda \geq 0$ and $\mu \geq 0$, with at least one of them nonzero, then we have

$$\operatorname{Re}\left[\lambda\frac{zf'(z)}{f(z)} + \mu\frac{s_n(z, f)}{f(z)}\right] > 0 \qquad (z \in E).$$

In Theorem 1 below we prove that this result continues to hold even when μ is a suitably restricted negative or complex number.

THEOREM 1. *Let $f \in S_t(1/2)$ and*

$$L = \operatorname{Re}\left[\lambda\frac{zf'(z)}{f(z)} + \mu\frac{s_n(z, f)}{f(z)}\right],$$

where $s_n(z, f)$ denotes the nth partial sum of f. Then $L > 0$ in E if (i) *$\lambda \geq 0$, $\mu \geq 0$ and at least one of them is nonzero,* (ii) *μ is a complex number and $\lambda > 4|\mu|$. The result is sharp in the sense that the ranges of λ and μ cannot be increased.*

PROOF. Case (i) being obvious, we take up the proof of (ii). Since f is given to be in $S_t(1/2)$ and $g(z) = z/(1 - z) \in K \subset S_t(1/2)$, it follows from Lemma 2 that if we choose $F(z) = \lambda/(1 - z) + \mu(1 - z^n)$ then the function

$$\begin{aligned}\frac{(f * gF)(z)}{(f * g)(z)} &= \frac{f(z) * zF(z)/(1 - z)}{f(z) * z/(1 - z)} \\ &= \frac{f(z) * [z/(1 - z)][\lambda/(1 - z) + \mu(1 - z^n)]}{f(z) * z/(1 - z)} \\ &= \lambda\frac{zf'(z)}{f(z)} + \mu\frac{s_n(z, f)}{f(z)},\end{aligned}$$

takes values in the convex hull of $F(E)$.

Now, since by hypothesis, $\lambda > 4|\mu|$, we find that, for $z \in E$,

$$\begin{aligned}\operatorname{Re} F(z) &= \operatorname{Re}\left[\frac{\lambda}{1-z} + \mu(1-z^n)\right] \\ &\geq \lambda/(1+r) - |\mu|(1+r^n) \\ &> \lambda/2 - 2|\mu| > 0\end{aligned}$$

(equality in the second line holds at $z = -|z| = -r$ when μ is negative and n is odd).

Since $(\lambda z f'(z)/f(z) + \mu s_n(z, f)/(f(z))$ takes values in the convex hull of $F(E)$, assertion (ii) now follows.

To prove that the ranges of λ and μ cannot be increased without violating the assertion of our theorem, we consider the function $f_0(z) = z/(1-z)$ which belongs to K and hence to $S_t(1/2)$. Let

$$\begin{aligned}L_0 &= \operatorname{Re}\left[\frac{\lambda z f_0'(z)}{f_0(z)} + \mu\frac{s_n(z, f_0)}{f_0(z)}\right] \\ &= \operatorname{Re}\left[\frac{\lambda}{1-z} + \mu(1-z^n)\right] \qquad (z \in E).\end{aligned}$$

If $\lambda \geq 0$, $\mu \geq 0$ with at least one of them nonzero, then clearly $L_0 > 0$ in E. On the other hand if $\lambda < 0$, then $L_0 \not> 0$ in E whatever μ may be. Finally, when $\lambda > 0$ and μ is negative, then it is seen that $L_0 > 0$ in E only when $(\lambda/2 - 2|\mu|) > 0$. This completes the proof of our theorem.

REMARK. Since the function $f_0(z)$ also belongs to K, Theorem 1 remains sharp within this subclass of $S_t(1/2)$.

The significance of the following theorem emerges from the fact that if $f \in K$, then $\operatorname{Re}(1/f'(z))$ need not be positive in the whole of the unit disc.

THEOREM 2. *If $f \in K$, then for all λ and μ with $\mu \geq 0$ and $\lambda > 2\mu$, we have*

$$\operatorname{Re}\left[\frac{\lambda f(z)}{z f'(z)} + \frac{\mu}{f'(z)}\right] > 0 \qquad (z \in E).$$

PROOF. Since $f \in K$ and the function $g(z) = z/(1-z)^2$ is in S_t, in view of Lemma 1 we conclude that for all $z \in E$ the function w, defined by

$$\begin{aligned}w(z) &= \frac{f(z) * [z/(1-z)^2][\lambda(1-z) + \mu(1-z)^2]}{f(z) * z/(1-z)^2} \\ &= \frac{\lambda f(z)}{z f'(z)} + \frac{\mu}{f'(z)},\end{aligned}$$

takes values in the convex hull of $F(E)$, where $F(z) = \lambda(1-z) + \mu(1-z)^2$.

Now, since by hypothesis, $\lambda > 2\mu$, $\mu \geq 0$, we find that

$$\begin{aligned}\operatorname{Re} F(z) &= \operatorname{Re}[\lambda(1-z) + \mu(1-z)^2] \\ &= (\lambda - 2\mu)(1 - r\cos\theta) + \mu[(1-r^2) + 2(1 - r\cos\theta)^2] \quad (z = re^{i\theta}) \\ &> 0, \qquad z \in E.\end{aligned}$$

The assertion of Theorem 2 is now clear.

The fact that for every $f \in S_t(1/2)$, $\operatorname{Re} f'(z) > 0$ only in $|z| < 1/\sqrt{2} = 0.707\ldots$ underlines the importance of our next theorem.

THEOREM 3. *If $f \in S_t(1/2)$, then*

$$\operatorname{Re}\left[\frac{f(z)}{z} + f'(z)\right] > 0$$

in $|z| < \rho = \sqrt{4\sqrt{2}-5} \doteqdot 0.81\ldots$. The number ρ is the best possible one.

PROOF. Consider the function

$$h(z) = \frac{1}{1-z} + \frac{1}{(1-z)^2} \qquad (z \in E). \tag{1}$$

We first proceed to prove that $\operatorname{Re} h(z) > 0$ in $|z| < \rho = \sqrt{4\sqrt{2}-5} \doteqdot 0.81\ldots$.

Letting $1/(1-z) = Re^{i\phi}$, we get

$$\frac{1}{1+r} \leq R \leq \frac{1}{1-r} \qquad (|z| = r) \tag{2}$$

and

$$\cos\phi = \frac{1+R^2-r^2R^2}{2R} \quad (\leq 1). \tag{3}$$

(1) and (3) provide

$$\begin{aligned} 2\operatorname{Re} h(z) &= 2[R\cos\phi + R^2\cos 2\phi] \\ &= 2 + (1-3r^2)t + (1-r^2)^2t^2 \qquad (t = R^2) \\ &= \psi(t), \text{ say.} \end{aligned}$$

It is now readily verified that for $r \geq \sqrt{7}-2$, t_1 given by $t_1 = (3r^2-1)/2(1-r^2)^2$ lies in the range of t $(= R^2)$ and that $\partial\psi/\partial t = 0$ and $\partial^2\psi/\partial t^2 > 0$ at $t = t_1$. We, therefore, conclude that for $r \geq \sqrt{7}-2$,

$$\min\psi(t) = \psi(t_1) = \frac{8(1-r^2)^2-(3r^2-1)^2}{4(1-r^2)^2} > 0,$$

if $r < \rho = \sqrt{4\sqrt{2}-5} \doteqdot 0.81\ldots$.

On the other hand, if $r < \sqrt{7}-2$, then one can easily see that

$$\min\psi(t) = \psi\left(\frac{1}{(1+r)^2}\right) = \frac{2(2+r)}{(1+r)^2} > 0.$$

To sum up, we have shown that

$$\operatorname{Re} h(z) > 0 \quad \text{in } |z| < \rho = \sqrt{4\sqrt{2}-5},$$

from which it follows that

$$\operatorname{Re} h(\rho z) > 0, \qquad z \in E. \tag{4}$$

Now taking $g(z) = z$ and $F(z) = \rho h(\rho z)$ in Lemma 2, we conclude that the values of the function

$$\begin{aligned} g(z) &= \frac{f(z) * z[\rho h(\rho z)]}{f(z) * z} \\ &= \frac{f(z) * z[\rho/(1-\rho z) + \rho/(1-\rho z)^2]}{f(z) * z} \\ &= \rho\left[\frac{f(\rho z)}{\rho z} + f'(\rho z)\right] \end{aligned}$$

lie in the convex hull of $F(E)$. However, in view of (4) we have

$$\operatorname{Re} F(z) = \operatorname{Re} \rho h(\rho z) > 0 \quad \text{in } E.$$

We have thus proved that

$$\operatorname{Re}\left[\frac{f(\rho z)}{\rho z} + f'(\rho z)\right] > 0$$

in E and hence

$$\operatorname{Re}\left[\frac{f(z)}{z} + f'(z)\right] > 0 \quad \text{in } |z| < \rho.$$

If we consider the function $f_0(z) = z/(1-z) \in K \subset S_t(1/2)$ then it is seen that

$$\frac{f_0(z)}{z} + f_0'(z) = \frac{1}{1-z} + \frac{1}{(1-z)^2},$$

and the assertion regarding the sharpness of the number ρ now becomes obvious in view of the definition of the function h.

THEOREM 4. *If $f \in K$, then*

$$\operatorname{Re}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) + \frac{1}{f'(z)}\right] > 0$$

in $|z| < \rho = (\sqrt{5}-1)/\sqrt{2} = 0.874\ldots$. The number ρ cannot be replaced by any larger one.

PROOF. Proceeding as in the proof of the previous theorem, one can show that the function h, defined by

$$h(z) = 2/(1-z) - 1 + (1-z)^2 \qquad (z \in E), \tag{5}$$

has the property that $\operatorname{Re} h(z) > 0$ only when $|z| < \rho = (\sqrt{5}-1)/\sqrt{2} = 0.874\ldots$, from which it follows that

$$\operatorname{Re} h(\rho z) > 0 \quad \text{in } E. \tag{6}$$

Since the function f is given to be in K and $g(z) = z/(1-\rho z)^2$ belongs to S_t, Lemma 1, in conjunction with (6), provides that the function p, defined by

$$\begin{aligned} p(z) &= \frac{f(z) * g(z)h(\rho z)}{f(z) * g(z)} \\ &= \frac{f(z) * [\rho z/(1-\rho z)^2][2/(1-\rho z) - 1 + (1-\rho z)^2]}{f(z) * \rho z/(1-\rho z)^2} \\ &= \left(1 + \frac{\rho z f''(\rho z)}{f'(\rho z)}\right) + \frac{1}{f'(\rho z)}, \end{aligned}$$

has positive real part in E. The desired conclusion is now obvious.

The sharpness of the number ρ follows from the fact that for the function $f_0(z) = z/(1-z) \in K$ we have

$$\begin{aligned} \left(1 + \frac{zf_0''(z)}{f_0'(z)}\right) + \frac{1}{f_0'(z)} &= \frac{2}{1-z} - 1 + (1-z)^2 \\ &= h(z) \quad (\text{given by } (5)), \end{aligned}$$

and that $\operatorname{Re} h(z) > 0$ only when $|z| < \rho = (\sqrt{5}-1)/\sqrt{2}$.

We observe that the disc $|z| < \rho = 0.874\ldots$ is much larger than the disc $|z| < \sqrt{2}/2 = 0.707$ in which $\operatorname{Re}(1/f'(z)) > 0$ for every $f \in K$.

We omit the proof of the following theorem.

THEOREM 5. *If $f \in S_t(1/2)$, then*

$$\operatorname{Re}\left[\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right] > 0$$

in $|z| < \rho = \sqrt{8\sqrt{2} - 11} = 0.56\ldots$. The number ρ is the best possible one.

If $f \in K$, then it is well known that $g(z) = (f(z) - f(-z))/2$ is an odd function in S_t and hence $\operatorname{Re}(z/g(z)) > 0$ in E. Our next theorem generalizes this latter result.

THEOREM 6. *Let $f \in K$. Then for each integer $n \geq 1$ we have*

$$\operatorname{Re} \frac{v_n(z,f) - v_n(-z,f)}{f(z) - f(-z)} > 0 \qquad (z \in E),$$

where $v_n(z,f)$ is the de la Valleé Poussin mean of f of order n.

PROOF. Let us first suppose that n is an odd integer, $n = 2m+1$, say, and consider the function F_{2m+1} defined by

$$(7)\ F_{2m+1}(z) = 2(1-z^2)\left[\frac{2m+1}{2m+2} + \frac{(2m+1)2m(2m-1)}{(2m+2)(2m+3)(2m+4)}z^2 + \frac{(2m+1)(2m)(2m-1)(2m-2)(2m-3)}{(2m+2)(2m+3)(2m+4)(2m+5)(2m+6)}z^4 + \cdots + \frac{(2m+1)(2m)\cdots 2\cdot 1}{(2m+2)(2m+3)\cdots(2(2m+1))}z^{2m}\right].$$

Obviously F_{2m+1} is regular in E (in fact it is an entire function), and we can write it in the form

$$(8)\ F_{2m+1}(z) = 2\left[\frac{2m+1}{2m+2} - \frac{2m+1}{(2m+2)}\left\{1 - \frac{2m(2m-1)}{(2m+3)(2m+4)}\right\}z^2 - \frac{(2m+1)2m(2m-1)}{(2m+2)(2m+3)(2m+4)}\left\{1 - \frac{(2m-2)(2m-3)}{(2m+5)(2m+6)}\right\}z^4 - \cdots - \frac{(2m+1)2m(2m-1)\cdots 3}{(2m+2)(2m+3)\cdots(2m+(2m-1))(4m)} \times\left\{1 - \frac{2\cdot 1}{(4m+1)(4m+2)}\right\}z^{2m} - \frac{(2m+1)2m\cdots 2\cdot 1}{(2m+2)(2m+3)\ldots(4m+2)}z^{2m+2}\right].$$

In view of (8) and (7) it is now easy to see that in E we have

$$\operatorname{Re} F_{2m+1}(z) \geq F_{2m+1}(|z|) > 0.$$

Next suppose that n is an even integer, $n = 2m$, say, and consider the function F_{2m} defined by

$$F_{2m}(z) = 2(1-z^2)\left[\frac{2m}{2m+1} + \frac{2m(2m-1)(2m-2)}{(2m+1)(2m+2)(2m+3)}z^2 + \cdots + \frac{2m(2m-1)\cdots 3\cdot 2z^{2m-2}}{(2m+1)(2m+2)\cdots(2m+(2m-1))}\right]$$

As before, one can see that $\operatorname{Re} F_{2m}(z) > 0$ in E.

In Lemma 1, letting $g(z) = z/(1-z^2)$, a function belonging to S_t, and

$$F(z) = \begin{cases} F_{2m+1}(z) & \text{if } n = 2m+1 \text{ is odd,} \\ F_{2m}(z) & \text{if } n = 2m \text{ is even,} \end{cases}$$

we conclude that for every integer $n \geq 1$, the function

$$w(z) = \frac{f(z) * zF(z)/(1-z^2)}{f(z) * z/(1-z^2)}$$

takes values in the right half-plane, that is, $\operatorname{Re} w(z) > 0$ in E.

A moderate calculation, however, shows that

$$w(z) = \frac{v_n(z,f) - v_n(-z,f)}{f(z) - f(-z)}.$$

This completes the proof of our theorem.

As observed earlier, if $f \in K$, then $g(z) = (f(z) - f(-z))/2$ is an odd starlike function. We conclude this paper with a theorem pertaining to g which, although not in tune with the earlier one, is of considerable intererst.

THEOREM 7. *If $f \in K$, then*

$$g(\rho z) \prec f(z)$$

in E, where

$$g(z) = \tfrac{1}{2}(f(z) - f(-z))$$

and $\rho = \sqrt{2} - 1 = 0.414\ldots$. The number ρ cannot be replaced by any larger one.

PROOF. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$ and

$$g(\rho z) = z + a_3\rho^3 z^3 + a_5 \rho^5 z^5 + \cdots,$$

the conclusion of our theorem would follow provided $\{\rho, 0, \rho^3, 0, \rho^5, \ldots\}$ is a subordinating factor sequence. In view of Lemma 3 this will be true if and only if

$$\operatorname{Re}\left[1 + 2\sum_{m=0}^{\infty} \rho^{2m+1} z^{2m+1}\right] = \operatorname{Re}\left[1 + \frac{2\rho z}{1-\rho^2 z^2}\right] > 0 \qquad (z \in E),$$

or,

$$1 - 2\rho/(1-\rho^2) \geq 0,$$

which is true by the choice of ρ.

To prove that the number ρ is the best possible one, let us consider the function $f(z) = z/(1-z) \in K$. It is then seen that for any $0 \leq \lambda \leq 1$, $g(\lambda z) = \lambda z/(1-\lambda^2 z^2)$. Since $g(-\lambda) = -\lambda/(1-\lambda^2) < -1/2$ if $\lambda > \sqrt{2} - 1$, from the fact that the range of f is the half-plane $\{w \mid \operatorname{Re} w > -1/2\}$ it follows that $g(\lambda z)$ cannot be subordinate to f in E if $\lambda > \sqrt{2} - 1 = \rho$.

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