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CHARACTERIZING S^m BY THE SPECTRUM OF THE LAPLACIAN ON 2-FORMS

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ABSTRACT. The Euclidean sphere S^{2n+1} is characterized by the spectrum of the Laplacian on 2-forms in all dimensions.

1. Introduction. It was recently shown [**4**, **5**] that within the class of Kaehler manifolds, complex projective n-space CP_n with the Fubini metric g_0 is characterized by the spectrum of the Laplacian on 2-forms in all dimensions. More precisely, let (M,g) be a compact Kaehler manifold with $\mathrm{Spec}^2(M,g) = \mathrm{Spec}^2(CP_n, g_0)$, where $\mathrm{Spec}^p(M,g)$ denotes the spectrum of the Laplacian with respect to the Kaehler metric g on p-forms of M. Then, (M,g) is holomorphically isometric to (CP_n, g_0) for all n. In this paper, we consider the problem of characterizing the constant curvature sphere S^m by the spectrum of its Laplacian on p-forms: If $\mathrm{Spec}^p(M,g) = \mathrm{Spec}^p(S^m, g_0)$ for some *fixed* p, is (M,g) isometric with (S^m, g_0), where g_0 is the constant curvature metric, that is, does there exist a diffeomorphism $f: M \to S^m$ such that $f^* g_0 = g$? The answer to this question is yes in the folowing cases:

(a) $p = 0$ and $m \leq 6$ [**1**, **8**];
(b) $p = 1$ and $m = 2, 3, 16, \ldots, 93$ [**9**];
(c) $p = 2$ and $m = 2, 3, 6, 7, 14, 17, \ldots, 178$ [**11**].

REMARK. Patodi [**7**] proved that if $\mathrm{Spec}^p(M,g) = \mathrm{Spec}^p(S^m, g_0)$ for $p = 0$ and 1, then (M,g) is isometric with (S^m, g_0) *in all dimensions.*

In order to obtain uniqueness for a fixed p in all (odd) dimensions we confine ourselves to the class of normal contact Riemannian manifolds and obtain the following statement.

THEOREM 1. *Let (M,g) be a compact normal contact Riemannian manifold. If $\mathrm{Spec}^2(M,g) = \mathrm{Spec}^2(S^{2n+1}, g_0)$, where g_0 is the metric of constant curvature $k = 1$, then g is a metric of the same constant curvature $k = 1$.*

2. The spectrum. Let (M,g) be a compact connected C^∞ Riemannian manifold without boundary, and with Laplacian $\Delta = -(dd^* + d^*d)$, where d is the operator of exterior differentiation and d^* is its adjoint with respect to the Riemannian metric g. Then, for each $p = 0, 1, 2, \ldots$, the spectrum of Δ is given by

$$\mathrm{Spec}^p(M,g) = \{0 \geq \lambda_{1,p} \geq \lambda_{2,p} \geq \cdots \geq \lambda_{k,p} \geq \cdots \downarrow -\infty\},$$

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each eigenvalue $\lambda_{i,p}$ repeated as often as its mutiplicity. By Hodge theory, $0 \in \mathrm{Spec}^p(M,g)$ if and only if the pth betti number $b_p(M)$ is not zero, and its multiplicity is then $b_p(M)$. For $p=2$, the Minakshisundaram-Pleijel-Gaffney formula is

$$\sum_{k=0}^{\infty} \exp(\lambda_{k,2}t) = \frac{1}{(4\pi t)^m} \sum_{i=0}^{N} a_{i,2}t^i + O(t^{N-m+1}), \qquad t \downarrow 0,$$

the coefficients $a_{i,2}$, $i=0,1,2$, being given by

$$a_{0,2} = \frac{m(m-1)}{2}V, \qquad V = \mathrm{vol}(M), \tag{2.1}$$

$$a_{1,2} = \frac{m^2-13m+24}{2}\int_M \rho\, dV, \tag{2.2}$$

$$\begin{aligned} a_{2,2} = \frac{1}{720}\int_M \big[2(m^2-31m+240)|R|^2 - 2(m^2-181m+1080)|S|^2 \\ + 5(m^2-25m+120)\rho^2\big]\, dV. \end{aligned} \tag{2.3}$$

where R, S, and ρ denote the curvature tensor, the Ricci tensor, and the scalar curvature of g, respectively, and $|R|^2 = \sum R_{ijkl}R^{ijkl}$, $|S|^2 = \sum R_{ij}R^{ij}$, R_{ijkl} and R_{ij} denoting the components of R and S, respectively [**7**].

If $\mathrm{Spec}^2(M,g) = \mathrm{Spec}^2(M',g')$, then $\dim M = \dim M'$, $V = V'$, $b_2(M) = b_2(M')$, and

$$\int_M \rho\, dV = \int_{M'} \rho'\, dV' \quad \text{and} \quad a_{2,2} = a'_{2,2}. \tag{2.4}$$

The following expression for $a_{2,2}$ will be useful:

$$720a_{2,2} = \int_M \left[Q_1|C|^2 + Q_2\left(|S|^2 - \frac{\rho^2}{m}\right) + Q_3\rho^2\right] dV, \tag{2.5}$$

where $|C|^2 = \sum C_{ijkl}C^{ijkl}$ is the square of the norm of the Weyl conformal curvature tensor. The components of C are

$$\begin{aligned} C_{ijkl} = R_{ijkl} - \frac{1}{m-2}(R_{jk}g_{il} - R_{jl}g_{ik} + g_{jk}R_{il} - g_{jl}R_{ik}) \\ + \frac{\rho}{(m-1)(m-2)}(g_{jk}g_{il} - g_{jl}g_{ik}), \end{aligned}$$

and

$$\begin{aligned} Q_1 &= 2(m-15)(m-16), \\ Q_2 &= \frac{8(m-15)(m-16)}{m-2} - 2(m^2-181m+1080), \\ Q_3 &= \frac{4(m-15)(m-16)}{m(m-1)} - \frac{2(m^2-181m+1080)}{m} + 5(m^2-25m+120). \end{aligned}$$

Note that $|S|^2 \geq \rho^2/m$ with equality holding if and only if g is an Einstein metric.

If (M', g') is a manifold of constant curvature k', then $|C'| = 0$ and $|S'|^2 = \rho'^2/m$, so by (2.5)

$$\int_M \left[Q_1|C|^2 + Q_2\left(|S|^2 - \frac{\rho^2}{m}\right) + Q_3\rho^2\right] dV = \int_{M'} Q_3\rho'^2\, dV'. \tag{2.6}$$

Thus, since Q_1, Q_2, and Q_3 are positive for $m = 3, 6, 7, 14, 17, 18, \ldots, 178$, and

$$\int_M \rho^2\, dV \geq \int \rho'^2\, dV',$$

the latter being a consequence of Schwarz's inequality and (2.4), g is a conformally flat Einstein metric. Hence, (M, g) is a manifold of constant curvature $k = k'$ in these dimensions [**11**]. For $m = 8$, Q_3 vanishes and Q_1 and Q_2 are both positive, so again g is a constant curvature metric. For $m = 15$ and 16, Q_1 vanishes and Q_2 and Q_3 are both positive, so g is an Einstein metric with scalar curvature ρ'. If $M' = S^m$ with the metric of constant curvature k', then, since $V = V'$ it follows from [**2**, p. 257] that (M, g) is isometric with (S^m, g') for $m = 15$ and 16. This extends Theorem 3.1 in [**11**].

The case $\rho =$ constant is interesting. For, since Q_1 and Q_2 are positive for $m = 9, \ldots, 13$, we may again conclude that g is a metric of constant curvature.

THEOREM 2. *Let (M, g) be a compact Riemannian manifold. If* $\mathrm{Spec}^2(M, g) = \mathrm{Spec}^2(S^m, g_0)$, *where g_0 is a metric of constant curvature k', then g is a metric of constant curvature $k = k'$ for $m = 2, 3, 6, 7, 8, 14, 15, 16, 17, \ldots, 178$. If, in addition, g is a metric of constant scalar curvature, then g is a metric of constant curvature $k = k'$ for $m = 2, 3, 6, \ldots, 178$.*

The case $m = 2$ is a consequence of the fact that $\mathrm{Spec}^2(M, g) = \mathrm{Spec}^2(S^2, g_0)$ implies $\mathrm{Spec}^0(M, g) = \mathrm{Spec}^0(S^2, g_0)$.

THEOREM 3. *Let (M, g) be a compact Riemannian manifold with* $\mathrm{Spec}^2(M, g) = \mathrm{Spec}^2(S^m, g_0)$, *where g_0 is a metric of constant curvature k', and for some $\lambda \in \mathbf{R}$, let*

$$\int_M (|S|^2 - \lambda p^2)\, dV = \int_{S^m} (|S'|^2 - \lambda\rho'^2)\, dV', \tag{2.7}$$

where the prime indicates corresponding quantities in (S^m, g_0). Then, if

(i) *$\lambda < 1/m$, g is a metric of constant curvature $k = k'$,*

(ii) *$\lambda \geq 1/m$, g is a metric of constant curvature $k = k'$ for each m satisfing $(\lambda - 1/m)Q_2 + Q_3 > 0$.*

PROOF. Since for $m = 15, 16$ the theorem follows from Theorem 2, we will assume that $m \neq 15, 16$ and therefore $Q_1 > 0$. Formula (2.6) may be rewritten in the form

$$\begin{aligned} \int_M \Bigg[& Q_1|C|^2 + \mu Q_2\left(|S|^2 - \frac{\rho^2}{m}\right) \\ & + (1-\mu)Q_2\left((|S|^2 - \lambda\rho^2) + \left(\lambda - \frac{1}{m}\right)\rho^2\right) + Q_3(\rho^2 - \rho'^2)\Bigg] dV = 0, \end{aligned} \tag{2.8}$$

where μ is a real number to be specified later. By (2.7) and the fact that $|S'|^2 = \rho'^2/m$, formula (2.8) becomes

$$\int_M \left[Q_1|C|^2 + \mu Q_2\left(|S|^2 - \frac{\rho^2}{m}\right) + \left(\left(\lambda - \frac{1}{m}\right)Q_2 + Q_3 - \left(\lambda - \frac{1}{m}\right)\mu Q_2\right)(\rho^2 - \rho'^2)\right] dV = 0. \tag{2.9}$$

If $\lambda < 1/m$, we specify μ as follows: Take μ to be of the same sign as Q_2 (note that $Q_2 \neq 0$ for any integer m), and $|\mu|$ to be so large that $(\lambda - 1/m)Q_2 + Q_3 - (\lambda - 1/m)\mu Q_2 > 0$. Since $|S|^2 - \rho^2/m \geq 0$ and $\int(\rho^2 - \rho'^2)\,dV \geq 0$, (2.9) implies that $C = 0$, $\rho = \rho'$, and $|S|^2 = \rho^2/m$. The last equality shows that g is Einstein, and so g is a metric of positive constant curvature.

If $\lambda \geq 1/m$, we take $\mu = 0$. Formula (2.9) then becomes

$$\int_M \left[Q_1|C|^2 + \left(\left(\lambda - \frac{1}{m}\right)Q_2 + Q_3\right)(\rho^2 - \rho'^2)\right] dV = 0.$$

Since m satisfies the inequality $(\lambda - 1/m)Q_2 + Q_3 > 0$, $C = 0$ and $\rho = \rho'$. Hence, by (2.7)

$$\begin{aligned}\int_M \left(|S|^2 - \frac{\rho^2}{m}\right) dV &= \int_M \left[(|S|^2 - \lambda\rho^2) + \left(\lambda - \frac{1}{m}\right)\rho^2\right] dV \\ &= \int_{S^m} \left[(|S'|^2 - \lambda\rho'^2) + \left(\lambda - \frac{1}{m}\right)\rho'^2\right] dV' \\ &= \int_{S^m} \left(|S'|^2 - \frac{\rho'^2}{m}\right) dV' = 0.\end{aligned}$$

But $|S|^2 \geq \rho^2/m$, so $|S|^2 = \rho^2/m$, that is, g is an Einstein metric. Since $C = 0$, g is a constant curvature metric.

3. Contact manifolds. An $m(= 2n + 1)$-dimensional C^∞ manifold is called a contact manifold if it carries a global 1-form η, called the contact form, with the property $\eta \wedge (d\eta)^n \neq 0$ everywhere. The classical example is the bundle of unit tangent vectors to an oriented $(n + 1)$-dimensional manifold. An odd-dimensional sphere S^{2n+1} possesses a contact structure which is not of this type. More generally, a smooth hypersurface of $(2n+2)$-dimensional affine space with the property that no tangent space contains the origin has a contact structure. J. Martinet showed that every compact 3-manifold carries a contact structure. A compact Hodge manifold B has a contact manifold canonically associated with it as a circle bundle with B as base space. Thus, the class of contact manifolds is quite extensive.

An almost contact structure (ϕ, X_0, η) on a $(2n + 1)$-dimensional C^∞ manifold M is given by a linear transformation field ϕ, a vector field X_0, and a 1-form η satisfying

$$\eta(X_0) = 1, \quad \phi X_0 = 0, \quad \text{and} \quad \phi^2 = -I + \eta \otimes X_0. \tag{3.1}$$

In this case, a Riemannian metric g can be found such that

$$\eta = g(X_0, \cdot) \quad \text{and} \quad g(\phi X, Y) = -g(X, \phi Y) \tag{3.2}$$

for any vector fields X and Y.

A contact manifold with contact form η has an underlying almost contact Riemannian structure (ϕ, X_0, η, g) such that $g(X, \phi Y) = d\eta(X, Y)$. If the almost complex structure J on $M \times R$ defined by

$$J(X, f d/dt) = (\phi X - f X_0, \eta(X) d/dt)$$

is integrable, the almost contact structure is said to be *normal*, and M is said to be a *normal contact manifold.* In this case, the unit vector field X_0 is a Killing vector field. Moreover,

$$g(R(X, X_0)Y, X_0) = g(X, Y) - \eta(X)\eta(Y) = g(\phi X, \phi Y) \tag{3.3}$$

and

$$S(X, X_0) = 2n\eta(X). \tag{3.4}$$

The sectional curvature of a plane section containing X_0 is therefore positive, and the Ricci curvature in the direction X_0 is $2n$.

The standard contact Riemannian structure on an odd-dimensional sphere is normal.

Put $\tilde{S}(X, Y) = S(X, \phi Y)$. Then $\tilde{S}$ is a skew symmetric bilinear form on M.

LEMMA 1. *Let M^{2n+1} be a compact normal contact Riemannian manifold with $b_2(M) = 0$. Then there exists a 1-form α on M such that $\tilde{S} = d\alpha$ and $\alpha(X_0) =$ const.*

PROOF. It is shown in [**6**] that $\tilde{S}$ is closed, so since $b_2(M) = 0$, $\tilde{S}$ is exact. We may therefore write $\tilde{S} = d\beta$. Let H denote the group of isometries of (M, g) preserving $\tilde{S}$. Then H is a compact Lie group. Let H_0 be the 1-parameter group of diffeomorphisms of M generated by X_0. Then, H_0 is a subgroup of H. Indeed, since X_0 is a Killing vector field, the elements of H_0 are isometries. In addition since $\tilde{S}(X_0, \cdot) = 0$, we obtain $L_{X_0}\tilde{S} = (i(X_0)d + di(X_0))\tilde{S} = 0$. Therefore, the elements of H_0 preserve $\tilde{S}$.

Set $\alpha = \int_H h^*(\beta)\, dh$, where h is an arbitrary element of H and dh is the invariant measure on H normalized by the condition $\int_H dh = 1$. Then,

$$d\alpha = \int_H h^*(d\beta)\, dh = \int_H h^*(\tilde{S})\, dh = \int_H \tilde{S}\, dh = \tilde{S}.$$

Clearly, $h^*(\alpha) = \alpha$ for any $h \in H$. Since $H_0 \subset H$ it follows that $L_{X_0}\alpha = 0$. Therefore, since $di(X_0)\alpha = L_{X_0}\alpha - i(X_0)d\alpha = 0$, we conclude that $i(X_0)\alpha =$ const.

PROOF OF THEOREM 1. Let (M, g) be a compact normal contact Riemannian manifold, and let (ϕ, X_0, η, g) be its underlying almost contact Riemannian structure. Set $\Phi(X, Y) = g(X, \phi Y)$. Then, Φ is skew symmetric. The following formulas are known (see [**3**]):

$$\nabla_X X_0 = -\phi X, \tag{3.5}$$

$$(\nabla_X \eta)(Y) = \Phi(X, Y), \tag{3.6}$$

$$(\nabla_X \phi)Y = g(X, Y)X_0 - \eta(Y)X, \tag{3.7}$$

$$(\nabla_X \Phi)(Y, Z) = \eta(Y)g(X, Z) - \eta(Z)g(X, Y) \tag{3.8}$$

for any vector fields X, Y, and Z.

Using (3.4)–(3.8) we obtain the following formulas by direct computation:

$$d^*\Phi = 2n\eta, \tag{3.9}$$

$$d^*\Phi^2 = 4(n-1)\eta \wedge \Phi, \tag{3.10}$$

$$i(\Phi)\tilde{S} = \tfrac{1}{2}(\rho - 2n), \tag{3.11}$$

$$i(\Phi^2)\tilde{S}^2 = \tfrac{1}{2}(\rho^2 - 2|S|^2 - 4n\rho + 12n^2), \tag{3.12}$$

$$i(\tilde{S})(\eta \wedge \Phi) = \tfrac{1}{2}(\rho - 2n)\eta, \tag{3.13}$$

where i is the adjoint of exterior multiplication, that is, if $\langle\ ,\ \rangle$ denotes the local scalar product with respect to the Riemannian metric g,

$$\langle i(\alpha)\beta, \gamma\rangle = \langle \beta, \alpha \wedge \gamma\rangle,$$

where α, β, and γ are forms of degrees p, q, and $q-p$, respectively. Denote by $(\alpha, \alpha') = \int_M \langle \alpha, \alpha'\rangle\, dV$ the global scalar product. By (3.9), (3.11), and Lemma 1

$$\begin{aligned}\frac{1}{2}\int_M (\rho - 2n)\, dV &= (i(\Phi)\tilde{S}, 1) = (\tilde{S}, \Phi) = (d\alpha, \Phi) = (\alpha, d^*\Phi)\\ &= 2n(\alpha, \eta) = 2n\alpha(X_0)V.\end{aligned}$$

On the other hand, by (2.4)

$$\frac{1}{2}\int_M (\rho - 2n)\, dV = \frac{1}{2}\int_{S^m} (\rho' - 2n)\, dV' = 2n^2 V$$

since $\rho' = 2n(2n+1)$. Therefore, $\alpha(X_0) = n$. Now, by (3.10), (3.12) and (3.13)

$$\begin{aligned}&\frac{1}{2}\int_M (\rho^2 - 2|S|^2 - 4n\rho + 12n^2)\, dV\\ &\quad = (i(\Phi^2)\tilde{S}^2, 1) = (\tilde{S}^2, \Phi^2)\\ &\quad = (d(\alpha \wedge d\alpha), \Phi^2) = (\alpha \wedge \tilde{S}, d^*\Phi^2)\\ &\quad = 4(n-1)(\alpha \wedge \tilde{S},\ \eta \wedge \Phi) = 4(n-1)(\alpha, i(\tilde{S})(\eta \wedge \Phi))\\ &\quad = 2(n-1)\alpha(X_0)\int_M (\rho - 2n)\, dV\\ &\quad = 8n^3(n-1)V.\end{aligned}$$

Therefore, $\int_M (\rho^2 - 2|S|^2)\, dV = 16n^3(n-1)V + \int_M (4n\rho - 12n^2)\, dV = (16n^4 - 4n^2)V$. On the other hand, for (S^m, g_0),

$$\int_{S^m} (\rho'^2 - 2|S'|^2)\, dV' = (16n^4 - 4n^2)V,$$

so we obtain

$$\int_M \left(|S|^2 - \frac{\rho^2}{2}\right) dV = \int_{S^m} \left(|S'|^2 - \frac{\rho'^2}{2}\right) dV'.$$

Applying Theorem 3 with $\lambda = \frac{1}{2}$ and noting that $(\frac{1}{2} - \frac{1}{m})Q_2 + Q_3 > 0$ for all odd $m \geq 3$, we conclude that g is a metric of constant curvature $k = 1$ for all $n \geq 1$.

We thank the referee for mentioning the following result due to Tsagas [**10**]: *For a given dimension m there exists an integer $p \in [0, m]$ such that if* $\mathrm{Spec}^p(M, g) = \mathrm{Spec}^p(S^m, g_0)$, *then g is a metric of the same constant curvature as g_0.*

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