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SPECTRAL ASYMPTOTICS FOR SPINOR LAPLACIANS AND MULTIPLICITIES

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ABSTRACT. We use Getzler's formula for the heat kernel of the spinor Laplacian to derive information about the asymptotic distribution of multiplicities in the quasi-regular representation of a semisimple Lie group G modulo a cocompact discrete subgroup Γ.

0. Introduction. Let G be a connected semisimple Lie group with finite center and let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. It is well known that the right regular representation R_Γ of G on $L^2(\Gamma\backslash G)$ splits into a countable direct sum of irreducible unitary representations and that for each class $\pi \in \hat{G}$ occurring in this decomposition the multiplicity $N_\Gamma(\pi)$ is finite. The integers $N_\Gamma(\pi)$ have been, and still are, the subject of a great deal of investigation. One direction consists in studying their distribution as π "approaches infinity" in $\hat{G}$. This is achieved by relating the $N_\Gamma(\pi)$ to the traces of suitable heat operators on $M = \Gamma\backslash G/K$ and then studying their asymptotic expansions. Here K is a maximal compact subgroup of G. Also, we shall assume for simplicity that Γ is torsion free and therefore M is a smooth compact manifold.

Thus, using the connection Laplacian on M naturally associated to a finite-dimensional representation τ of K, Wallach proves in [**7**] that

(0.1)
$$\sum_{\pi\in\hat{G}} N_\Gamma(\pi)\langle\pi_K,\tau\rangle e^{t\chi_\pi(\Omega)} = (4\pi t)^{-m/2}\dim(\tau)\operatorname{vol}(M) + o(t^{-m/2}) \quad \text{as } t\to 0^+,$$

where $\langle\pi_K,\tau\rangle$ is the intertwining number of π restricted to K and τ, χ_π is the infinitesimal character of π, Ω stands for the Casimir element of G, $m=\dim(M)$ and $\operatorname{vol}(M)$ is the volume of M (with respect to a natural Riemannian structure).

In fact, one has an asymptotic expansion of the form

$$\sum_{\pi\in\hat{G}} N_\Gamma(\pi)\langle\pi_K,\tau\rangle e^{t\chi_\pi(\Omega)} \sim \sum_{k=0}^{\infty} a_k(\tau)t^{k-m/2} \quad \text{as } t\to 0^+; \tag{0.2}$$

but, except when $\operatorname{rank}(G/K)=1$ (see [**4**]), not much is known about the coefficients $a_k(\tau)$, $k>0$.

In this note we show that for $0\le k\le m/2$ the coefficients a_k, viewed as functions of τ, satisfy a series of equations which can be regarded as asymptotic analogues of the alternating sum formulas for multiplicities in [**5**]. This information is extracted from a limit formula of Getzler for the heat kernel of the Dirac operator (cf. [**2, 3**]).

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1. A consequence of Getzler's formula. Given an inner product space V over $\mathbf{R}$ we denote by $C(V)$ the associated Clifford algebra, defined as in [**2**], and by $\sigma\colon C(V) \to \bigwedge V$, $\theta\colon \bigwedge V \to C(V)$ the canonical isomorphisms (of vector spaces) betwen the Clifford algebra and the exterior algebra of V, with $\theta = \sigma^{-1}$. According to whether $m = \dim M$ is even or odd, $\mathbf{C}(V) = C(V) \otimes_{\mathbf{R}} \mathbf{C}$, resp $\mathbf{C}^{\mathrm{ev}}(V) = \theta(\bigwedge_{\mathbf{C}}^{\mathrm{ev}} V)$, is a simple algebra over $\mathbf{C}$ and thus has a unique simple module $S(V)$ of dimension $2^{m/2}$, resp. $2^{[(m+1)/2]}$, called the space of spinors over V. In what follows we shall identify $\mathbf{C}(V)$, resp. $\mathbf{C}^{\mathrm{ev}}(V)$, with $\operatorname{End} S(V)$. We recall that $S(V)$ can be equipped with an inner product such that each $v \in V$ acts as a selfadjoint operator on $S(V)$. We shall need the following elementary result, whose proof is similar to that of Theorem 1.8 in [**2**] and will be omitted.

LEMMA. *Let $\omega \in \bigwedge_{\mathbf{C}}^{2l} V$ and $h \in \operatorname{End} S(V)$. Then*

$$\operatorname{Tr}(\theta(\omega)h) = (-1)^l \dim S(V) \langle \sigma(h), \overline{\omega} \rangle.$$

We now consider a compact spin manifold M, of dimension m, and denote by $S(TM)$ its spin bundle. $\operatorname{End} S(TM)$ will be identified with $\mathbf{C}(TM)$ if m is even, respectively with $\mathbf{C}^{\mathrm{ev}}(TM)$ when m is odd. We shall also identify TM and T^*M via the Riemannian metric. The above isomorphisms σ and θ will then induce bundle isomorphisms $\sigma\colon \mathbf{C}(TM) \to \bigwedge_{\mathbf{C}} T^*M, \theta\colon \bigwedge_{\mathbf{C}} T^*M \to \mathbf{C}(TM)$.

Let E be a Hermitian vector bundle M, with connection ∇^E. The Dirac operator D_E, acting on the space of C^∞ sections $\Gamma(S(TM) \otimes E)$, is defined in terms of a local orthonormal frame $\{e_1, \ldots, e_m\}$ by the expression

$$D_E = \sum_{j=1}^{m} \theta(e_j) \otimes I(\nabla^S_{e_j} \otimes I + I \otimes \nabla^E_{e_j}),$$

where ∇^S is the connection on $S(TM)$ induced by the Riemannian connection on TM. With the present definition of the Clifford algebra, $D_E$is skew-adjoint and therefore the associated heat semigroup is $e^{tD_E^2}$.

Let $K_t(x,y)$ be the Schwartz kernel of $e^{tD_E^2}$. For each $x \in M$, $K_t(x,x)$ is an endomorphism of $S(T_xM) \otimes E_x$ and so $\operatorname{Tr}_{E_x} K_t(x,x) \in \operatorname{End} S(T_xM)$. We now define a (nonhomogeneous) form $k_t(x) = \sum_{j=0}^m k_t^{(j)}(x)$ on M by

$$k_t(x) = \sigma(\operatorname{Tr}_{E_x} K_t(x,x)).$$

Getzler's main result in [**3**] implies that

$$\lim_{\varepsilon \to 0^+} \sum_{j=0}^{m} \varepsilon^{m-j} k^{(j)}_{\varepsilon^2 t}(x) = (4\pi t)^{-m/2} \operatorname{ch}(tR_E)(x) \wedge \hat{A}(tR)(x), \tag{1.1}$$

where $R \in \Gamma(\bigwedge^2 T^*_{\mathbf{C}}M \otimes \operatorname{End} TM)$ is the Riemannian curvature of M, $R_E \in \Gamma(\bigwedge^2 T^*_{\mathbf{C}}M \otimes \operatorname{End} E)$ is the curvature of the connection ∇^E,

$$\hat{A}(R) = \det\left(\frac{R/2}{\sinh(R/2)}\right)^{1/2} \in \Gamma\left(\bigwedge^{\mathrm{ev}} T^*_{\mathbf{C}}M\right),$$

and

$$\operatorname{ch}(R_E) = \operatorname{Tr}(e^{R_E}) \in \Gamma\left(\bigwedge^{\mathrm{ev}} T^*_{\mathbf{C}}M\right).$$

Taking in both sides of (1.1) the scalar product with $\overline{\omega}(x)$, where ω is a $2l$-form on M, one obtains

$$\lim_{t\to 0^+} \varepsilon^{m-2l}\langle k_{\varepsilon^2 t}(x), \overline{\omega}(x)\rangle = (4\pi t)^{-m/2} t^l \langle \mathrm{ch}(R_E)(x) \wedge \hat{A}(R)(x), \overline{\omega}(x)\rangle.$$

Now, due to the above lemma, one can rewrite this as

$$\text{(1.2)}\quad \lim_{t\to 0^+} t^{m/2-l}\,\mathrm{Tr}((\theta(\omega)(x)I)K_t(x,x) = (-1)^l C_m \langle \mathrm{ch}(R_E)(x) \wedge \hat{A}(R)(x), \overline{\omega}(x)\rangle,$$

where

$$C_m = 2^{-[(m+1)/2]}\pi^{-m/2}.$$

Since $x \longmapsto (\theta(\omega)(x) \otimes I)K_t(x,x) \in \mathrm{End}\, S(T_xM) \otimes E_x$ is the diagonal restriction of the Schwartz kernel of the operator $(\theta(\omega) \otimes I)e^{tD_E^2}$, by integrating (1.2) over M one obtains the following statement.

PROPOSITION. *For any $2l$-form ω on M, one has*

$$\text{(1.3)}\qquad \lim_{t\to 0^+} t^{m/2-l}\,\mathrm{Tr}((\theta(\omega) \otimes I))e^{tD_E^2} = (-1)^l C_m \int_M \mathrm{ch}(R_E) \wedge \hat{A}(R) \wedge *\omega.$$

Let us note that for the 0-form $\omega = 1$ this is just Weyl's formula for spinor Laplacians, while for $l = m/2$ (m even) and $i^{-l}\omega =$ the volume form, one gets the index formula for Dirac operators.

2. Application to multiplicities. We shall now specialize the above result to the case of a compact locally symmetric space $M = \Gamma\backslash G/K$, where G, K and Γ are as in the introduction. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K) and let $\mathfrak{p}$ be the orthogonal of $\mathfrak{k}$ with respect to the Cartan-Killing form B. We endow M with the Riemannian metric obtained from $B|\mathfrak{p} \times \mathfrak{p}$ via the identification of $\mathfrak{p}$ with the tangent space at $o = 1 \cdot K$ to G/K.

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$ and let $\mathfrak{h}$ be the centrlizer of $\mathfrak{t}$ in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Φ be the root system of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ and let Φ_k be the root system of $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$. We fix, once and for all, a set of positive roots Ψ_k for Φ_k and a set of positive roots Ψ for Φ, in a compatible fashion. As usual, we denote by ρ, ρ_k, and ρ_n the half-sum of the roots in Ψ, Ψ_k, and $\Psi_n = \Psi - \Psi_k$ respectively. Let T be the maximal torus of K with Lie algebra $\mathfrak{t}$. The dual group $\hat{T}$ will be identified, via exponentiation, to a lattice $L_T \subset i\mathfrak{t}^*$.

The spin module associated to $(\mathfrak{p}, \langle\ ,\ \rangle)$ will be denoted S. It is, in particular, a $\mathfrak{k}$-module. We do not postulate the existence of a G-invariant spin structure on the symetric space G/K, and thus S need not be a K-module. Consider, however, an irreducible $\mathfrak{k}$-module V_ν whose highest weight $\nu \in i\mathfrak{t}^*$ satisfies the condition

$$\text{(2.1)}\qquad \nu + \rho_n \in L_T.$$

Since every weight of S differs from ρ_n by a sum of roots, (2.1) is easily seen to guarantee the fact that the representation of $\mathfrak{k}$ on $S \otimes V_\nu$ lifts to a representation of K. This representation of K, in turn, gives rise first to a homogeneous bundle over G/K and then, by passing to Γ-orbits, to a vector bundle over M which will be denoted $S(M,\nu)$. This bundle comes equipped with a Hermitian structure, induced by the K-invariant inner product on $S \otimes V_\nu$ and with a unitary connection $\nabla^{(S,\nu)}$, inherited from that of the principal bundle $G \to G/K$ (defined by the splitting

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$). We can therefore form the twisted Dirac operator $D_\nu \colon \Gamma(S(M,\nu)) \to \Gamma(S(M,\nu))$. Explicitly, after identifying $\Gamma(S(M,\nu))$ with the subspace $(C^\infty(\Gamma/G)\otimes S \otimes V_\nu)^K$ of all K-invariant elements in $C^\infty(\Gamma/G)\otimes S\otimes V_\nu$, D_ν is given by the formula

$$D_\nu = \sum_{j=1}^{m} R_\Gamma(X_j)\otimes\theta(X_j)\otimes I, \tag{2.2}$$

where $\{X_1,\dots,X_m\}$ is an orthonormal basis for $\mathfrak{p}$. Moreover (see [**6**], §3),

$$-D_\nu^2 = -R_\Gamma(\Omega)\otimes I\otimes I + (\|\nu+\rho_k\|^2 - \|\rho\|^2)I\otimes I\otimes I, \tag{2.3}$$

where Ω is the Casimir element of $\mathfrak{g}$.

We recall that $\bigwedge \mathfrak{p}_{\mathbf{C}}^*$ (resp. $\bigwedge^{\mathrm{ev}} \mathfrak{p}_{\mathbf{C}}^*$, if m is odd) and $\mathrm{End}(S)$ are isomorphic as $\mathrm{SO}(\mathfrak{p})$-modules. In particular $\omega \in \wedge^{\mathrm{ev}}\mathfrak{p}_{\mathbf{C}}^*$ is K-invariant if and only if $\theta(\omega)\in \mathrm{End}_K(S)$.

Consider now a unitary representation π of G on a Hilbert space $\mathfrak{H}_\pi$. Then

$$I\otimes\theta(\omega)\otimes I \colon \mathfrak{H}_\pi\otimes S\otimes V_\nu \to \mathfrak{H}_\pi\otimes S\otimes V_\nu$$

commutes with the (tensor product) action of K, and therefore restricts to an operator

$$\theta_{\pi,\nu}(\omega) \colon (\mathfrak{H}_\pi\otimes S\otimes V_\nu)^K \to (\mathfrak{H}_\pi\otimes S\otimes V_\nu)^K,$$

where the superscript K signifies passage to K-invariant elements. Note that if π is irreducible, $(\mathfrak{H}_\pi\otimes S\otimes V_\nu)^K$ has finite dimension.

Let $r\in \bigwedge^2 \mathfrak{p}_{\mathbf{C}}^*\otimes \mathrm{End}(\mathfrak{p})$ and $r_\nu \in \bigwedge^2 \mathfrak{p}_{\mathbf{C}}^*\otimes\mathrm{End}(V_\nu)$ be defined as follows

$$r = -\sum_{1\le i,j\le m} \mathrm{ad}[X_i,X_j]\otimes\xi_i\wedge\xi_j,$$
$$r_\nu = -\sum_{1\le i,j\le m} \tau_\nu[X_i,X_j]\otimes\xi_i\wedge\xi_j,$$

where $\{X_1,\dots,X_m\}$ is an orthonormal basis of $\mathfrak{p}$, $\{\xi_1,\dots,\xi_m\}$ is it dual basis for $\mathfrak{p}^*$, and τ_ν denotes the representation of K on V_ν. It is easy to check that r and r_ν are independent of the orthonormal basis, and also that they are K-invariant. We then form

$$\hat{A}(r) = \det\left(\frac{r/2}{\sinh(r/2)}\right)^{1/2} \in \left(\bigwedge^{\mathrm{ev}} \mathfrak{p}_{\mathbf{C}}^*\right)^K$$

and

$$\mathrm{ch}(r_\nu) = \mathrm{Tr}_{V_\nu}(e^{r_\nu}) \in \left(\bigwedge^{\mathrm{ev}} \mathfrak{p}_{\mathbf{C}}^*\right)^K.$$

Finally, we denote

$$\hat{G}_{\Gamma,\nu} = \{\pi\in\hat{G};\, N_\Gamma(\pi)>0,\, \dim(\mathfrak{H}_\pi\otimes S\otimes V_\nu)^K>0\}$$

and recall that χ_π stands for the infinitesimal character of $\pi\in\hat{G}$.

THEOREM. *For each* $\omega\in(\bigwedge^{2l}\mathfrak{p}_{\mathbf{C}}^*)^K$, *one has*

$$\lim_{t\to0^+} t^{m/2-l}\sum_{\pi\in\hat{G}_{\Gamma,\nu}} N_\Gamma(\pi)\,\mathrm{Tr}\,\theta_{\pi,\nu}(\omega)e^{\chi_\pi(\Omega)t} = (-1)^l C_m\langle \mathrm{ch}(r_\nu)\wedge\hat{A}(r),\overline{\omega}\rangle\,\mathrm{vol}(\Gamma\backslash G/K). \tag{2.4}$$

PROOF. Let us assume for the moment that the $\mathfrak{k}$-action on S does lift to K and therefore M is a spin manifold. $S(TM)$ is then the bundle induced by S. We denote by E_ν the bundle on M induced by V_ν and by $\tilde{\omega}$ the form on M whose lift to G/K is the G-invariant form determined by ω. Clearly, $\theta(\tilde{\omega}) \in \operatorname{End} S(TM)$ is just the indomorphism induced by $\theta(\omega) \in \operatorname{End}_K(S)$. Furthermore, under the identification of $\Gamma(S(TM) \otimes E_\nu)$ with $(C^\infty(\Gamma\backslash G) \otimes S \otimes V_\nu)^K$, the mulitplication operator $\theta(\tilde{\omega}) \otimes I$ becomes

$$\theta_{R_\Gamma,\nu}(\omega) = I \otimes \theta(\omega) \otimes I|(L^2(\Gamma\backslash G) \otimes S \otimes V_\nu)^K.$$

Let us also note that, since the Riemannian connection on TM coincides with that induced by the canonical G-invariant connection on $T(G/K)$, the Dirac operator D_{E_ν} is precisely the operator D_ν given by (2.2).

From the decomposition of R_Γ into irreducible components,

$$L^2(\Gamma\backslash G) = \sum_{\pi\in\hat{G}}^{\oplus} N_\Gamma(\pi)\mathfrak{H}_\pi,$$

it follows that

$$(L^2(\Gamma\backslash G) \otimes S \otimes V_\nu)^K \cong \sum_{\pi\in\hat{G}_{\Gamma,\nu}}^{\oplus} N_\Gamma(\pi)(\mathfrak{H}_\pi \otimes S \otimes V_\nu)^K.$$

So, each $\pi \in \hat{G}_{\Gamma,\nu}$ contributes $N_\Gamma(\pi)$ summands of the form $(\mathfrak{H}_\pi \otimes S \otimes V_\nu)^K$; in turn, each of these summands is invariant under both D_ν^2 dand $\theta_{R_\Gamma,\nu}(\omega)$. Moreover, one has:

$$D_\nu^2|(\mathfrak{H}_\pi \otimes S \otimes V_\nu)^K = (\chi_\tau(\Omega) - (\|\nu+\rho_k\|^2 - \|\rho\|^2))I \quad \text{(cf. (2.3))}$$

and

$$\theta_{R_\Gamma,\nu}(\omega)|(\mathfrak{H}_\pi \otimes S \otimes V_\nu)^K = \theta_{\pi,\nu}(\omega).$$

Therefore,

$$\operatorname{Tr}(\theta_{R_\Gamma,\nu}(\omega)e^{tD^2}) = e^{(\|\rho\|^2-\|\nu+\rho_k\|^2)t} \sum_{\pi\in\hat{G}_{\Gamma,\nu}} N_\Gamma(\pi) \operatorname{Tr}\theta_{\pi,\nu}(\omega)e^{\chi_\pi(\Omega)t},$$

and thus (2.4) follows immediately from (1.3).

Finally, let us drop the assumption that S integrates to a representation of K. The role of the bundle $S(TM) \otimes E$ is then assumed by $S(M,\nu)$, the bundle induced by the K-module $S \otimes V_\nu$. Now $\operatorname{End} S(M,\nu)$ is induced by $\operatorname{End}(S \otimes V_\nu) = \operatorname{End}(S) \otimes \operatorname{End}(V_\nu)$ and, unlike S and V_ν which are only $\mathfrak{k}$-modules, both $\operatorname{End}(S)$ and $\operatorname{End}(V_\nu)$ are in fact K-modules. With this observation it is easy to see that the arguments in §1 still apply, giving (1.3) and therefore (2.4). □

When $\omega = 1$, (2.4) gives no new information beyond that coming from (0.1). At the opposite extreme, when $l = m/2$ (assuming m is even) and $i^{-l}\omega =$ the invariant volume form v on G/K, the left-hand side of (2.4) is independent of t and coincides with the index of D_ν. Thus (see also [**1**, §1]), (2.4) becomes

$$\sum_{\pi\in\hat{G}_{\Gamma,\nu}} N_\Gamma(\pi)(\dim(\mathfrak{H}_\pi \otimes S^+ \otimes V_\nu)^K - \dim(\mathfrak{H}_\pi \otimes S^- \otimes V_\nu)^K)$$
$$= C_m i^{m/2}\langle(\operatorname{ch}(r_\nu) \wedge \hat{A}(r))^{(m)}, v\rangle \operatorname{vol}(\Gamma\backslash G/K),$$

which is essentially Miatello's alternating sum formula for multiplicities, associated to the homogeneous symbol of D_ν (cf. [4]). In general, (2.4) can be viewed as interpolating between the two extremes.

Let us finally interpret the theorem in terms of the coefficient functions of the asymptotic expansion (0.2). Let

$$S = \sum_{i=1}^{N}{}^{\oplus} S_i, \qquad \sigma = \sum_{i=1}^{N}{}^{\oplus} \sigma_i$$

be the decomposition of S, as a $\mathfrak{k}$-module, into irreducible components, and let P_i denote the orthogonal projection in $\operatorname{End}_K(S)$ associated to $S_i, 1 \le i \le N$. Then

$$\theta(\omega) = \sum_{i=1}^{N}{}^{\oplus} c^i(\omega) P_i$$

with

$$c^i(\omega) = \operatorname{Tr}(\theta(\omega)P_i)/\dim S_i.$$

Hence

$$\theta_{\pi,\nu}(\omega) = \sum_{i=1}^{N}{}^{\oplus} c^i(\omega) I \otimes P_i \otimes I | (\mathfrak{H}_\pi \otimes S \otimes V_\nu)^K$$

and therefore

$$\operatorname{Tr} \theta_{\pi,\nu}(\omega) = \sum_{i=1}^{N} c^i(\omega) \langle \pi_K^*, \sigma_i \otimes \tau_\nu \rangle.$$

In fact each irreducible constituent σ_i of the spin representation σ is known to be of the form $\tau_{s\rho-\rho_k}$, where $s \in W^1 = \{w \in W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}); w\Psi \supset \Psi_k\}$. We can thus rewrite the last equality as

$$\operatorname{Tr} \theta_{\pi,\nu}(\omega) = \sum_{s \in W^1} c_s(\omega) \langle \pi_k^*, \tau_{s\rho-\rho_k} \otimes \tau_\nu \rangle,$$

where now

$$c_s(\omega) = \sum_{i \in I_s} c^i(\omega), \quad \text{with } I_s = \{i; \sigma_i = \tau_{s\rho-\rho_k}\}.$$

Using (2.5) and also the fact that $N_\Gamma(\pi^*) = N_\Gamma(\pi)$, $\chi_{\pi^*}(\Omega) = \chi_\pi(\Omega)$, the left-hand side of (2.4) becomes

$$\lim_{t \to 0^+} t^{m/2-l} \sum_{s \in W^1} c_s(\omega) \sum_{\pi \in \hat{G}_{\Gamma,\nu}} N_\Gamma(\pi) \langle \pi_K, \tau_{s\rho-\rho_k} \otimes \tau_\nu \rangle e^{\chi_\pi(\Omega)t}.$$

Thus, the information given by the theorem for the coefficients of the negative powers of t in the asymptotic expansion (0.2) amounts to the following family of equations.

COROLLARY. *For each $\omega \in (\bigwedge^{2l} \mathfrak{p}_{\mathbf{C}}^*)^K$, one has*

$$\sum_{s \in W^1} c_s(\omega) a_j(\tau_{s\rho-\rho_k} \otimes \tau_\nu) = 0, \quad \text{if } 0 \le j \le l-1,$$

and

$$\sum_{s \in W^1} c_s(\omega) a_l(\tau_{s\rho-\rho_k} \otimes \tau_\nu) = (-1)^l C_m \langle \operatorname{ch}(r_\nu) \wedge \hat{A}(r), \overline{\omega} \rangle \operatorname{vol}(\Gamma \backslash G/K).$$

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