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A REMARK ON A_1-WEIGHTS FOR THE STRONG MAXIMAL FUNCTION

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ABSTRACT. If g is a locally integrable function and M is the Hardy-Littlewood maximal function then $(Mg)^\delta$ represents an A_1-weight of M for every $0 < \delta < 1$; that is, $M(Mg)^\delta(x) \leq C_\delta(Mg(x))^\delta$ a.e. In this paper we show that this result does not hold in general if we replace M by the strong maximal operator.

Given an operator T defined on some subclass X of the set of measurable functions in $\mathbf{R}^n$ we say that $w \in X$ is an A_1-weight for T if

$$|Tw(x)| \leq Cw(x) \quad \text{a.e.},$$

where C is a constant which depends on w. We denote the class of all A_1-weights of T by $A_1(T)$. (See Muckenhoupt [**3**].)

The theory of weights has been very valuable in the study of many operators in classical harmonic analysis. Among others, the following result has been particularly important since it gives a description of the class $A_1(T)$, when $T = M$ is the Hardy-Littlewood maximal operator (i.e., the operator given by the supremum on the averages on cubes which contain a fixed point).

Let g be locally integrable and so that $Mg(x) < \infty$, a.e. Then $w = (Mg)^\delta \in A_1(M)$ for every $0 \leq \delta < 1$. (See Coifman and Rochberg [**1**].) Moreover, as an easy consequence of the reverse Hölder inequality one also has a converse statement. Namely, *if $w \in A_1(M)$, there exist $h(x)$ in L^1_{loc}, $\delta > 0$, and $k(x)$ with k and $1/k \in L^\infty$, such that $w(x) = k(x)(Mh(x))^\delta$.* So that, indeed, all A_1-weights of M are essentially of the form $(Mg)^\delta$.

In this paper we show that the first part of the above result does not hold in general if we replace M by the strong maximal operator M_S (averages on "intervals"). The study of whether or not this could be the case was proposed to the author by J. L. Rubio de Francia, to whom we are indebted. The question appears in [**2**] as an open problem.

We state now the main result. Further generalizations to the setting of product domains are given below.

PROPOSITION. *There exists $g \in \bigcap_{0<p\leq\infty} L^p(\mathbf{R}^n)$ with the following property. For every $0 < \delta$ and every constant $C > 0$, there is a set S of positive measure such that*

$$M_S(M_S g)^\delta(x) \geq C(M_S g)^\delta(x), \qquad \forall x \in S. \tag{1}$$

Indeed, inequality (1) *holds on a set of infinite measure.*

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For simplicity we will only consider here the case $n = 2$, the other cases being similar. This proposition is an easy consequence of the following.

LEMMA. *$\forall N \in \mathbf{N}$, there exists $v = v_N$ with compact support so that*

$$\{M_S(M_S v)^\delta(x)\}^{1/\delta} \geq c_\delta N M_S v(x), \qquad \forall x \in [0,1]^2.$$

Moreover,

$$\|v_N\|_{L^p} \leq c_p 2^N, \quad \text{for } 0 < p \leq \infty. \tag{2}$$

In order to see how this lemma implies the above result, one can just consider the sequence of functions $\{v_N\}$ whose existence is ensured by the lemma and form the function

$$g(x) = \sum 2^{-2N} v_N(x - x_N),$$

where the x_N's are chosen very distant one from another in such a way that the values of Mg on the square $[x_N, x_N+1]^2$ are the same as the values of $M(2^{-2N}v_N)$ on $[0,1]^2$. This can be easily done due to the compactness of the support of the v_N's. In fact, a simple analysis of the constants appearing in the proof of the lemma shows that $X_N = (2^N, 2^N)$ would do it.

PROOF OF THE LEMMA. Given $k, j \in \mathbf{Z}$, let $Q_{k,j}$ denote the unit square with lower left corner at the point (k,j). For $k = 1, 2, \ldots, N$ let d_k be the integral part of $2^k/k$ and set

$$v(x) = v_N(x) = \sum_{1 \leq k \leq N} 2^k \chi_{Q_{k,d_k}}(x).$$

Then clearly $\|v\|_\infty = 2^N$ and $\|v\|_{L^p} = \sum_1^N 2^{kp}$ for $0 < p < \infty$. Therefore, (2) holds with $c_p = 2(2^p - 1)^{-1/p}$.

Now, if R_k is the rectangle $[1, k+1] \times [1, d_k + 1]$, a simple computation shows that $\forall x \in [0,1]^2$ we have

$$M_S v(x) \leq M_S v((1,1)) = \sup_{1 \leq k \leq N} |R_k|^{-1} \int_{R_k} v(y)\, dy \leq 2.$$

On the other hand, if $\tilde{R}_k = [k, k+1] \times [1, d_k + 1]$ then for every $x \in \tilde{R}_k$ we have

$$M_S v(x) \geq |\tilde{R}_k|^{-1} \int_{\tilde{R}_k} v(y)\, dy = 2^k/d_k \geq k/2.$$

Hence, $f = M_S v(x) \geq k/2$ on $Q_{k,j}$ for $j = 1, 2, \ldots, N$ provided that $d_k \geq N$ (e.g., if $k \geq 2\log_2 N$).

Finally, if we take the *square* $R = [0, N+1] \times [0, N+1]$ we obtain for every $x \in [0,1]^2$

$$\begin{aligned}
[M_S(M_S v)^\delta(x)]^{1/\delta} &\geq [M_S(f)^\delta((0,0))]^{1/\delta} \\
&\geq \left(|R|^{-1} \int_R (f)^\delta\, dy\right)^{1/\delta} \\
&\geq c \left[(N+1)^{-2} \sum_{2\log_2 N \leq k \leq N} N(k)^\delta\right]^{1/\delta} = O(N).
\end{aligned}$$

This finishes the proof of the lemma and, therefore, of the above proposition. Q.E.D.

REMARKS. (1) The fact that in the last stage of the previous proof we have considered a square shows that the function $w = (M_S g)^\delta$ is not even an A_1-weight for the Hardy-Littlewood maximal operator. This is stronger than $w \notin A_1(M_S)$ since $Mf \leq M_S f \ \forall f$ and, therefore, $A_1(M_S) \subset A_1(M)$.

(2) One could try to produce A_1-weights for the operators arising in product domains as follows: Let M_i, $i = 1, 2$, be the 1-dimensional Hardy-Littlewood operator in the x_i-direction. Given $f \in L^1_{\text{loc}}$ we consider $(M_2 M_1 f)^\delta$ for some $\delta < 1$ and we ask whether this is necessarily a weight in $A_1(M_2 \circ M_1)$. The answer is no and the function g is again a counterexample.

In fact, the weaker statement $M_1(M_2 f)^\delta \leq C(M_2 M_1 f)^\delta$ a.e. is not true in general. For if we take $N \in \mathbf{N}$ and $v = v_N$ as in the lemma then

$$M_2 M_1 v(x) \sim 1 \quad \text{on } Q_{1,1} = [1,2]^2,$$

whereas

$$(M_1(M_2 v)^\delta(x))^{1/\delta} \sim N \quad \text{on } Q_{1,1}.$$

Perhaps it is interesting to notice here that the inequality

$$M_2(M_1 f)^\delta \leq (M_2 M_1 f)^\delta \qquad \forall f$$

is trivial for $0 < \delta \leq 1$, by Jensen's inequality.

(3) Another possible way of constructing A_1-weights in the setting of product domains could be to look at $(M_2(M_1 f)^\delta)^\mu$, $0 < \delta, \mu < 1$, for some $f \in L^1_{\text{loc}}$. Again, this is not the case. To see this, it suffices to consider $f(x) = (g(x))^{1/\delta}$, for $0 < \delta < 1$ fixed.

A simple computation shows that $(M_2(M_1(v_N)^{1/\delta})^\delta)^\mu \sim N^{(1-\delta)\mu}$ on $Q_{1,1}$, whereas from remark (2) one has

$$M_1(M_2(M_1(v_N)^{1/\delta})^\delta)^\mu \geq M_1(M_2 v_N)^\mu \sim N^\mu \quad \text{on } Q_{1,1}.$$

Therefore, $w = (M_2(M_1 f)^\delta)^\mu \notin A_1(M_1)$ (hence $w \notin A_1(M_S) \cup A_1(M_1 \circ M_2) \cup A_1(M_2 \circ M_1) = A_1(M_S)$). However, w does belong to $A_1(M_2)$, from Coifman and Rochberg's result.

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