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ON SEQUENCES WITHOUT WEAK* CONVERGENT CONVEX BLOCK SUBSEQUENCES

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ABSTRACT. Let X be a Banach space such that X^* contains a bounded sequence without a weak* convergent convex block subsequence. Then, subject to Martin's Axiom and the negation of the Continuum Hypothesis, X contains $l_1(\mathfrak{c})$. With the same assumption, every nonreflexive Grothendieck space has l_∞ as a quotient.

1. Introduction. In this paper, we continue the study of the relationship between weak* sequential convergence in the dual of a Banach space X and the existence in X of isomorphic copies of l_1 or $l_1(\Gamma)$. It is, of course, clear that if X contains $l_1(\mathfrak{c})$ where $\mathfrak{c} = 2^\omega$, the cardinal of the continuum, then X^* contains a bounded sequence with no weak* convergent subsequence. On the other hand, the existence of such a sequence in X^* does not imply that X contains $l_1(\omega_1)$ [**7**], or even l_1 [**6**]. Hagler and Johnson [**5**] showed that if X^* contains an infinite-dimensional subspace Y in which all weak* convergent sequences are norm convergent, then X does contain l_1. Bourgain [**2**] refined this result by proving that if X does not contain l_1 then every bounded sequence in X^* has a weak* convergent convex block subsequence. This last theorem was rediscovered by Rosenthal [**10**], whose illuminating proof was based on a lemma, which we reproduce here as 3A.

The question of whether the result of Hagler and Johnson, or the Bourgain-Rosenthal refinement, can be extended to give a copy of $l_1(\omega_1)$ in X was answered in part by Talagrand [**11**]. We recall that a Banach space X is said to be a *Grothendieck space* if every weak* convergent sequence in X^* is weakly convergent. There was a conjecture that every nonreflexive Grothendieck space should contain l_∞, or at least should have l_∞ as a quotient. An example given in [**8**] showed that such spaces do not necessarily contain l_∞. Talagrand showed that if the Continuum Hypothesis (CH) is true then there exists an infinite compact space K such that $\mathcal{C}(K)$ is a Grothendieck space and such that no quotient of $\mathcal{C}(K)$ is isomorphic to l_∞. Since it is true in general that X has l_∞ as a quotient if and only if X has a subspace isomorphic to $l_1(\mathfrak{c})$, Talagrand's CH example does not contain $l_1(\omega_1)$. To see that this example satisfies the hypothesis of the Hagler-Johnson theorem, we may take $Y = l_1(K)$, regarded as the subspace of atomic measures of $\mathcal{M}(K) = \mathcal{C}(K)^*$.

We shall show in this paper that, subject to Martin's Axiom and the negation of the Continuum Hypothesis (in fact, subject to a set-theoretic assumption weaker than this), the Hagler-Johnson-Bourgain-Rosenthal result can be extended to give

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an embedding of $l_1(\mathfrak{c})$ in X. With the same assumption, we show that every nonreflexive Grothendieck space contians $l_1(\mathfrak{c})$, and so has l_∞ as a quotient. Our methods are based on Rosenthal's Lemma 3A, from which we deduce an unconditional result about $C(K)$ spaces. The other important tool we use is a theorem of Argyros, Bourgain, and Zachariades [**1**]: Let K be a compact space and let μ be a finite measure on K. Let κ be a cardinal greater than ω_1. Let X be a subspace of $C(K)$ containing a bounded family $(x_\alpha)_{\alpha<\kappa}$ with the property that $\|x_\alpha - x_\beta\|_{L_1(\mu)} > \eta > 0$ for all $\alpha \neq \beta$. Then X contains $l_1(\kappa)$. In fact there exists $\Lambda \subset \kappa$ with $\#\Lambda = \kappa$ and $(x_\alpha)_{\alpha\in\Lambda}$ equivalent to the unit vector basis of $l_1(\Lambda)$.

Our notation and terminology are mostly standard. The first infinite cardinal which is just the set of all natural numbers is written ω; ω_1 is the first uncountable cardinal; $\mathfrak{c} = 2^\omega$ is the continuum; the special cardinal $\mathfrak{p}$ is introduced in §2. If (f_n) is a linearly independent sequence in a vector space, we say that (g_n) is a *convex block subsequence* of (f_n) if there are finite subsets B_n of ω and nonnegative real numbers α_j such that, for all n,

$$\max B_n < \min B_{n+1},$$
$$g_n = \sum_{j\in B_n} \alpha_j f_j, \quad \text{and} \quad \sum_{i\in B_n} \alpha_j = 1.$$

We say that (g_n) is a *rational convex block subsequence* of (f_n) if, in addition, the α_j's are all rational. We write $(g_n) \prec (f_n)$ if (g_n) is *eventually* a convex block subsequence of (f_n), that is to say, that there exists $N \in \omega$ such that $(g_n)_{n=N}^\infty$ is a convex block subsequence of (f_n).

2. Diagonalization and the cardinal $\mathfrak{p}$. Following [**4**], we write $\mathfrak{p}$ for the largest cardinal having the property:

> If $\kappa < \mathfrak{p}$ and $(M_\alpha)_{\alpha<\kappa}$ is a family of subsets of ω with $\bigcap_{\alpha\in F} M_\alpha$ infinite for all finite $F \subset \kappa$, then there exists an infinite $M \subset \omega$ with $M\backslash M_\alpha$ finite for all $\alpha < \kappa$.

Informally speaking, we can diagonalize fewer than $\mathfrak{p}$ finitely compatible infinite subsets of ω. It is known that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$; Martin's Axiom implies that $\mathfrak{p} = \mathfrak{c}$, and the relative consistency of $\omega_1 < \mathfrak{p} = \mathfrak{c}$ has been established. Further facts about $\mathfrak{p}$ and its relationship with Martin's Axiom can be found in [**4**].

The present paper is not the first place in which the cardinal $\mathfrak{p}$ has been involved in considerations of weak* sequential convergence and l_1 subspaces. Indeed Fremlin has pointed out (page 56 of [**4**]) that a theorem of Figiel, Ghoussoub, and Johnson [**3**] can be conveniently stated using this cardinal: if E is an order continuous Banach lattice and F is a subspace of E such that the ball (F^*) is not weak* sequentially compact, then F contains $l_1(\mathfrak{p})$.

We need to note that we can diagonalize fewer than $\mathfrak{p}$ compatible rational (convex) block subsequences of a given sequence of vectors. The following lemma uses an entirely standard way of coding finite sequences of rationals as integers.

2A LEMMA. *Let κ be a cardinal with $\kappa < \mathfrak{p}$. Let $(f_n)_{n=1}^\infty$ be a linearly independent sequence of vector space and let $(g_n^\alpha)_{n=1}^\infty$ be rational convex block subsequences of $(f_n)_{n=1}^\infty$ for $\alpha < \kappa$. Assume that for each finite $F \subset \kappa$ there exists $(g_n)_{n=1}^\infty$ which is a rational convex block subsequence of each (g_n^α) for $\alpha \in F$. Then there* *exists* $(h_n)_{n=1}^{\infty}$ *such that, for each* $\alpha < \kappa, (h_n)$ *is eventually a rational convex block subsequence of* (g_n^{α}).

PROOF. Fix enumerations $(q_n)_{n=1}^{\infty}$ of $\mathbb{Q}_+$, the positive rationals, and $(p_n)_{n=1}^{\infty}$ of the positive primes. Let $q_0 = 0$. For

$$g = \sum_{j=1}^{n} q_{k(j)} f_j \in \mathbb{Q} - \operatorname{conv}\{f_j : j \in \omega\}$$

define

$$\pi(g) = \prod_{j=1}^{n} p_j^{k(j)}.$$

When $\underset{\sim}{g} = (g_n)$ is a rational convex block subsequence of (f_n), define

$$M(\underset{\sim}{g}) = \{\pi(h) : h \in \mathbb{Q} - \operatorname{conv}\{g_n : n \in \omega\}\} \subset \omega.$$

Consider the family of all subsets of ω which consist of all the sets $M(\underset{\sim}{g}^{\alpha})$ $(\alpha < \kappa)$, together with all the sets

$$M_n = \{s \in \omega : s \text{ is not divisible by } p_m \ (m < n)\}.$$

This family is finitely compatible, since if F is a finite subset of κ and $N \in \omega$, we may choose (g_n) as in the statement of the lemma and then note that

$$\pi(g_n) \in M_N \cap \bigcap_{\alpha \in F} M(\underset{\sim}{g}^{\alpha}) \qquad \text{for all } n \geq N.$$

Since $\kappa < \mathfrak{p}$, there is an infinite subset M of ω which is almost contained in each member of this family. Using the fact that M has nonempty intersection with each of the M_n's we can construct a rational convex *block* subsequence (h_n) of (f_n), with the property that $\pi(h_n) \in M_n$ for all n.

For any $\alpha < \kappa$, $\pi(h_n) \in M(\underset{\sim}{g}^{\alpha})$ for all but finitely many n, and hence $(h_n) \prec (g_n^{\alpha})$. $\square$

3. The main result. We start by quoting the following lemma from [**10**].

3A LEMMA (H. ROSENTHAL). *Let A be a set and let (f_n) be a uniformly bounded sequence of real-valued functions on A. Assume that no convex block subsequence of (f_n) converges everywhere on A. Then there exists $c \in \mathbb{R}$, $\delta > 0$, and a convex block subsequence (g_n) of (f_n) such that*

(i) $\sup_{x \in A} \operatorname{osc}_{n \to \infty} g_n(x) = 2\delta$, *and*

(ii) *for every convex block subsequence (h_n) of (g_n) and every $\eta < \delta$, there exists $x \in A$ with*

$$\limsup_{n \to \infty} h_n(x) > c + \eta \quad \textit{and} \quad \liminf_{n \to \infty} h_n(x) < c - \eta.$$

3B THEOREM. *Let K be a compact space, let A be a subset of ball $\mathcal{C}(K)$ and let (λ_n) be a sequence in ball $\mathcal{M}(K)$. Assume that, for every convex block subsequence (μ_n) of (λ_n), there exists $x \in A$ such that $\langle \mu_n, x \rangle$ fails to converge as $n \to \infty$. Then there exists $\eta > 0$, a positive measure $\mu \in \mathcal{M}(K)$ and a family $(x_\alpha)_{\alpha < \kappa}$ in A such that $\int_K |x_\alpha - x_\beta| \, d\mu > \eta$ for all distinct $\alpha, \beta < \kappa$.*

PROOF. First note that we can assume all the λ_n are positive. For if (λ_n^+) has a convex block subsequence (μ_n), $\mu_n = \sum_{m \in B_n} \alpha_m \lambda_m^+$, which converges everywhere on A, then no convex block subsequence of (μ'_n) can converge everywhere on A where $\mu'_n = \sum_{m \in B_n} \alpha_m \lambda_m^-$.

We now apply Lemma 3A and obtain $(\mu_n) \prec (\lambda_n)$, $c \in \mathbb{R}$, and $\eta > 0$ such that, for every $(\nu_n) \prec (\mu_n)$, there exists $x \in A$ with

$$\limsup_{n\to\infty} \langle \nu_n, x\rangle > c + \eta \quad \text{and} \quad \liminf_{n\to\infty} \langle \nu_n, x\rangle < c - \eta.$$

The next stage is to construct, by transfinite induction, a family $(\underset{\sim}{\mu}^\alpha)_{\alpha<\mathfrak{p}}$ of rational convex block subsequences of $\underset{\sim}{\mu} = (\mu_n)$ and a family $(x^\alpha)_{\alpha<\mathfrak{p}}$ in A such that:

(i) $\underset{\sim}{\mu}^\beta \prec \underset{\sim}{\mu}^\alpha$ whenever $\alpha < \beta < \mathfrak{p}$.

(ii) $\langle \mu_n^\beta, x^\alpha\rangle \to c$ as $n \to \infty$ whenever $\alpha < \beta < \mathfrak{p}$.

(iii) $\langle \mu_{2n}^\beta, x^\beta\rangle > c + \eta$ and $\langle \mu_{2n+1}^\beta, x^\beta\rangle < c - \eta$ for all n and $\beta < \mathfrak{p}$.

To start the construction, we choose $x^0 \in A$ such that

$$\limsup_{n\to\infty} \langle \mu_n, x^0\rangle > c + \eta \quad \text{and} \quad \liminf_{n\to\infty} \langle \mu_n, x^0\rangle < c - \eta$$

and then choose $\mu_k^0 = \mu_{n(k)}$ in such a way that (iii) is satisfied with $\beta = 0$.

Now supose $\gamma < \mathfrak{p}$ and that the constuction has been carried out for $\beta < \gamma$. If γ is a limit ordinal, we may use Lemma 2A and the fact that $\gamma < \mathfrak{p}$ to show that there exists $\underset{\sim}{\nu}$ such that $\underset{\sim}{\nu} \prec \underset{\sim}{\nu}^\beta$ for all $\beta < \gamma$. We now choose $x^\gamma \in A$ and a subsequence $(\mu_k^\gamma) = (\nu_{n(k)})$ so that (iii) is satisfied for $\beta = \gamma$. The validity of (i) and (ii) for $\alpha < \beta = \gamma$ follows from the inductive hypothesis and the fact that $\underset{\sim}{\mu}^\gamma \prec \underset{\sim}{\mu}^\beta$ for all $\beta < \gamma$. If γ is a successor ordinal, $\gamma = \beta + 1$ say, we start by defining ν_n to be a rational convex combination of μ_{2n}^β and μ_{2n+1}^β in such a way that $\langle \nu_n, x^\beta\rangle \to c$ as $n \to \infty$. We then continue by choosing x^γ and $\mu_k^\gamma = \nu_{n(k)}$ as in the case of a limit ordinal.

By conditions (ii) and (iii), whenever $\alpha < \beta$, we have

$$\int |x^\alpha - x^\beta|\, d\mu_n^\beta \geq |\langle \mu_n^\beta, x^\beta\rangle - \langle \mu_n^\beta, x^\alpha\rangle| > \eta \quad \text{for all large enough } n.$$

Consequently, $\int |x^\alpha - x^\beta|\, d\nu \geq \eta$ for all ν in the nonempty weak* compact set

$$K_\beta = \bigcup_{m\in\omega} w^* \operatorname{cl}\operatorname{conv}\{\mu_n^\beta : n \geq m\}.$$

By (i) $K_\alpha \supseteq K_\beta$ when $\alpha < \beta < \mathfrak{p}$, and if μ is any member of $\bigcap_{\beta<\mathfrak{p}} K_\beta$ we have $\int |x^\alpha - x^\beta|\, d\mu \geq \eta$ for all distinct $\alpha, \beta < \mathfrak{p}$. □

3C COROLLARY. *If K is compact and in $\mathcal{M}(K)$ there is a bounded sequence (λ_n), with no weak* convergent block subsequence, then $L_1(\mu)$ is nonseparable for some $\mu \in \mathcal{M}(K)$. In fact K carries a measure μ with $L_1(\mu)$ isometric to $L_1(\{0,1\}^{\mathfrak{p}})$.*

PROOF. The theorem shows that for some $\mu \in \mathcal{M}(K)$, $L_1(\mu)$ has density character at least $\mathfrak{p}$. This implies the second assertion, using 2.1 of [**7**] for instance. □

3D THEOREM. *Assume that $\mathfrak{p} > \omega_1$. If X is a Banach space and in ball X^* there is a sequence with no weak* convergent convex block subsequence, then X contains an isomorphic copy of $l_1(\mathfrak{p})$.*

PROOF. By embedding X in $\mathcal{C}(K)$ for suitable K and taking $A = \operatorname{ball}(X) \subseteq \mathcal{C}(K)$ we are in a position to apply Theorem 3B. This gives us $\mu \in \mathcal{M}(K)^+$ and $(x^\alpha)_{\alpha<\mathfrak{p}} \subseteq$

$X \subseteq L_\infty(\mu)$ with $\|x^\alpha\|_\infty = 1$ and $\|x^\alpha - x^\beta\|_{L_1(\mu)} > \eta > 0$ for distince $\alpha, \beta < \mathfrak{p}$. By the theorem of Argyros, Bourgain, and Zacharides, there exists $\Lambda \subseteq \mathfrak{p}$ with $\#\Lambda = \mathfrak{p}$ such that $(x^\alpha)_{\alpha\in\Lambda}$ is equivalent, for the L_∞-norm, to the unit vector basis of $l_1(\Lambda)$. Thus X contains $l_1(\mathfrak{p})$ as claimed. □

3E COROLLARY. *Assume $\mathfrak{p} = \mathfrak{c} > \omega_1$. A Banach space X contains an isomorphic copy of $l_1(\mathfrak{c})$ if and only if X^* contains a bounded sequence with no weak* convergent convex block subsequence.*

3F COROLLARY. *If $\mathfrak{p} > \omega_1$, then every nonreflexive Grothendieck space has a subspace isomorphic to $l_1(\mathfrak{p})$.*

If $\mathfrak{p} = \mathfrak{c} > \omega_1$, then every nonreflexive Grothendieck space has a quotient isomorphic to l_∞.

PROOF. It suffices to show that X^* contains a bounded sequence with no weak* convergent convex block subsequence. This is clear since X is Grothendieck and X^* must contain l_1 (by Rosenthal's theorem [**9**]). □

REMARK. The first-named author has recently obtained the following unconditional result. If X^* contains an infinite-dimensional subspace in which weak* sequential convergence and norm convergence coincide, then X^* contains $L_1(\{0,1\}^{\mathfrak{p}})$.

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