PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 1, May 1987

ALGEBRA DIRECT SUM DECOMPOSITION OF $C_R(X)$. II

R. D. MEHTA AND M. H. VASAVADA

ABSTRACT. We give conditions under which the sets of constancy of a closed subalgebra of $C_R(X)$, which is complemented by a closed subalgebra, are retracts of X.

Introduction. Let X be a compact Hausdorff space and let $C_R(X)$ denote the Banach algebra of all real-valued continuous functions on X with supremum norm. Let B be a closed subalgebra of $C_R(X)$ with $1 \notin B$. Then $Z(B)$, the set of common zeros of elements of B, is not empty [**6**, Lemma 1.1]. We say that B is complemented by an algebra if there exists a closed subalgebra A of $C_R(X)$ with $1 \in A$ such that $A \oplus B = C_R(X)$. Such a decomposition of $C_R(X)$ has been considered in [**3**, **5**, and **6**]. If B is a closed ideal of $C_R(X)$ and if it is complemented by an algebra, then it is well known that $Z(B)$ is a retract of X [**8**]. The following example [**5**] shows that this result is not true if B is not an ideal.

EXAMPLE 1. Let $X = [-1, 1]$ and $B = \{g \in C_R[-1, 1] \colon g \text{ is even and } g(1) = 0\}$. Then $C_R(X) = A \oplus B$, where $A = \{f \in C_R[-1, 1] \colon f \text{ is constant on } [0, 1]\}$. Thus B is complemented by an algebra but $Z(B) = \{-1, 1\}$ is not a retract of X.

In our main result we give conditions under which $Z(B)$ is a retract if B is complemented by an algebra. If fact, we show that under the given conditions, each set of constancy of B is a retract. This generalizes the result regarding a complemented ideal.

If A is a subalgebra of $C_R(X)$ and $x, y \in X$, we say that x is A-related to y, and write $x \overset{A}{-} y$, if $f(x) = f(y)$ for every f in A. This is an equivalence relation on X and a subset of X is an equivalence class for this relation if and only if it is a set of constancy of A. If A and B are subalgebras of $C_R(X)$ and $x, y \in X$, we say that there is an (A, B) chain from x to y if there is an ordered set of points $\{x_0, x_1, \ldots, x_n, x_{n+1}\}$ with $x_0 = x$, $x_{n+1} = y$ such that either $x_i \overset{A}{-} x_{i+1} \overset{B}{-} x_{i+2}$ or $x_i \overset{B}{-} x_{i+1} \overset{A}{-} x_{i+2}$ $(i = 0, 1, 2, \ldots, n-1)$. An (A, B) chain of the form $\{x, x_1, x_2, \ldots, x_n, x\}$ where $x \neq x_1, \ldots, x_i \neq x_{i+1}, \ldots, x_n \neq x$ is called an (A, B) round trip.

Sets of constancy of a complemented algebra. We observe that in Example 1, the number of nontrivial sets of constancy of B is infinite. We shall show that when B has only a finite number of nontrivial sets of constancy, each set of constancy of B is a retract, if B is complemented by an algebra. Henceforth, for brevity, we shall call a set of constancy of a closed subalgebra C of $C_R(X)$, a C-set.

Received by the editors March 5, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46J10.
Key words and phrases. Set of constancy, subalgebra, retract.

Also in the sequel A and B will denote closed subalgebras of $C_R(X)$ with $1 \in A$ and $1 \notin B$.

LEMMA 2. *Let $A \oplus B = C_R(X)$. If F_1, F_2 are A-sets and K_1, K_2 are B-sets such that $F_i \cap K_j \neq \emptyset$ $(i, j = 1, 2)$, then $F_1 = F_2$ or $K_1 = K_2$.*

PROOF. Suppose, if possible, that F_1, F_2, K_1, K_2 are all distinct. Since $A + B$ separates points of X, $F_i \cap K_j$ is a singleton $(i, j = 1, 2)$. Let $F_i \cap K_j = \{x_{ij}\}$. Since F_1, F_2, K_1, K_2 are distinct, $F_1 \cap F_2 = \emptyset$ and $K_1 \cap K_2 = \emptyset$. Hence $x_{11}, x_{12}, x_{21}, x_{22}$ are distinct. But then $x_{11} \overset{B}{-} x_{21} \overset{A}{-} x_{22} \overset{B}{-} x_{12} \overset{A}{-} x_{11}$ is an (A, B) round trip, which is not possible by [**4**, Proposition 2]. Hence F_1, F_2, K_1, K_2 ar not distinct. Since $F_i \cap K_j \neq \emptyset$ $(i, j = 1, 2)$, we must have $F_1 = F_2$ or $K_1 = K_2$.

LEMMA 3. *Let K be a closed subset of X and F be the union of those A-sets whose intersection with K is nonempty. Then F is closed.*

PROOF. Since the decomposition of X determined by A is upper semicontinuous [**7**, Exercise 7.5.7(F)], the result follows by Exercise 5.2.9(I) of [**7**].

We are now ready to prove our main result.

THEOREM 4. *If $A \oplus B = C_R(X)$, and if only a finite number of B-sets are nontrivial, then each B-set is a retract.*

PROOF. Let $K_1, K_2, \ldots, K_n$ be distinct nontrivial B-sets. We shall show that K_1 is a retract. The proof for showing that K_i $(i = 2, 3, \ldots, n)$ is a retract will be analogous.

For $i = 1, 2, \ldots, n$, let F_i denote the union of all those A-sets whose intersection with K_i is nonempty. By Lemma 3, each F_i is closed. To prove that K_1 is a retract, we shall show that the ideal $I_{K_1} = \{g \in C_R(X) \colon g|_{K_1} = 0\}$ is complemented by an algebra A_1 in $C_R(X)$ [**8**].

Let S be a fixed A-set such that $S \subset F_1$. We now define A_1 as follows.

(a) Suppose that $F_i \cap K_1 = \emptyset$ $(i = 2, 3, \ldots, n)$. Then let

$$A_1 = \{f \in C_R(X) \colon f \text{ is constant on } F_2 \cup F_3 \cup \cdots \cup F_n \cup S \text{ and } f \text{ is constant on each } A\text{-set } F \text{ contained in } F_1\}.$$

(b) Suppose that $F_i \cap K_1 \neq \emptyset$ for some $i \geq 2$. Without loss of generality, we assume that $F_i \cap K_1 \neq \emptyset$ for $i = 2, 3, \ldots, m$ and $F_i \cap K_1 = \emptyset$ for $i = m + 1, \ldots, n$. Let $H = \bigcup_{i=m+1}^{n} F_i$. In view of Lemma 2, there exists only one A-set S_i such that $S_i \subset F_i$ and $S_i \cap K_1 \neq \emptyset$ $(i = 2, 3, \ldots, m)$. Some of these S_i's may coincide. Let $S_{j_1}, S_{j_2}, \ldots, S_{j_r}$ be the distinct S_i's and let E_t denote the union of those F_i's $(i = 2, 3, \ldots, m)$ which contain S_{j_t} $(1 \leq t \leq r)$. Take A_1 to be the algebra of those functions in $C_R(X)$ which are constant on each of E_t $(1 \leq t \leq r)$, constant on $H \cup S$, and constant on each A-set F where $F \subset F_1$.

A_1 as defined above is a closed algebra, $1 \in A_1$ and each A_1-set intersects K_1 in precisely one point. Hence $A_1 \oplus I_{K_1} = C_R(X)$ [**1**, Corollary 4.10]. It follows that K_1 is a retract of X.

COROLLARY 5. *If $A \oplus B = C_R(X)$ and if only finitely many A-sets are nontrivial, then each of them is a retract. (Recall that $1 \in A$ and $1 \notin B$.)*

PROOF. Let F be a fixed A-set. Define $A_1 = \{f \in A \colon f = 0 \text{ on } F\}$ and $B_1 = \{g \in C_R(X) \colon g \text{ is constant on each } B\text{-set}\}$. Then $C_R(X) = B_1 \oplus A_1$ and the result follows from Theorem 4, since a closed set is an A_1-set if and only if it is an A-set.

REMARKS 6. (1) We do not know whether the conclusion of Theorem 4 holds if the number of nontrivial B-sets is countably infinite.

(2) Let X be a compact metric space and let D be an upper semicontinuous decomposition of X with a finite number of nontrivial members. Then no member of D can be a limit set, so that $D^{(1)} = 0$ (see [**2**] for definitions). Hence if A_D is the algebra of all functions on $C_R(X)$ which are constant on each member of D, by [**2**, Theorem 1.9], A_D is complemented by a closed subspace. On the other hand, by Corollary 5, A_D will not be complemented by a closed subalgebra unless all members of D are retracts.

(3) We prove below (Example 7) that B need not be complemented by an algebra if the number of nontrivial B-sets is finite and each B-set is a retract. Interestingly, Theorem 4 is useful in proving this negative result.

(4) Examples can be given to show that if B is complemented by an algebra and each B-set is a retract, then also the number of nontrivial B-sets need not be finite.

EXAMPLE 7. Let $X = \{z \in \mathbf{C}\colon \frac{1}{2} \leq |z| \leq 1\}$, $K_1 = \{z \in \mathbf{C}\colon |z| = \frac{1}{2}\}$, $K_2 = \{z \in \mathbf{C}\colon |z| = 1\}$ and let $B = \{g \in C_R(X)\colon g$ is constant on K_1 and zero on $K_2\}$. Then K_1 and K_2 are the only nontrivial B-sets and they are retracts of X. We shall show, however, that B is not complemented by an algebra. To see this, suppose, if possible, that $A \oplus B = C_R(X)$, where A is a closed subalgebra with $1 \in A$. Take $D = \{z \in \mathbf{C}\colon |z| \leq 1\}$, $D_1 = \{z \in \mathbf{C}\colon |z| \leq \frac{1}{2}\}$ and define $A_1 = \{\varphi \in C_R(D)\colon \varphi|_X \in A\}$, $B_1 = \{\psi \in C_R(D)\colon \psi$ is constant on D_1 and $\psi = 0$ on $K_2\}$. Then A_1 and B_1 are closed subalgebras of $C_R(D)$ with $1 \in A_1$, $1 \notin B_1$, and $C_R(D) = A_1 \oplus B_1$. Hence by Theorem 4, $K_2 = Z(B_1)$ is a retract of D, which is not true. Hence B is not complemented by an algebra in $C_R(X)$. B is, however, complemented by a closed subspace (see Remark 6(2)).

With the help of Theorem 4, we give one more condition under which A-sets and B-sets are retracts when $A \oplus B = C_R(X)$ (Corollary 9).

THEOREM 8. *Let $C_R(X) = A \oplus B$. Then X is a finite union of A-sets and B-sets if and only if both A and B have only a finite number of nontrivial sets of constancy.*

PROOF. Suppose that A and B have only a finite number of nontrivial sets of constancy. Let Y denote the union of nontrivial A-sets. If $Y = X$, we are done. If not, let $p \in X - Y$. Suppose that $p \notin Z(B)$. Since $A \oplus B = C_R(X)$, by Theorem 2.1 of [**6**], there exists an (A, B) chain from p to some z in $Z(B)$. Hence p must be A-related or B-related to some $q \neq p$. Since $p \notin Y$, $p \overset{A}{-} q$ is not possible. Hence $p \overset{B}{-} q$ and p belongs to some nontrivial B-set. It follows that $X - Y$ is contained in the union of nontrivial B-sets and $Z(B)$. This proves the required result one way.

To prove the converse, suppose that $X = F_1 \cup F_2 \cup \cdots \cup F_m \cup K_1 \cup K_2 \cup \cdots \cup K_n$ where F_1's are A-sets $(1 \leq i \leq m)$ and K_j's are B-sets $(1 \leq i \leq n)$. Let E be a nontrivial A-set, $E \neq F_i$ $(1 \leq i \leq m)$. Then $E \subset \bigcup_{j=1}^{n} K_j$, as $E \cap F_i = \varnothing$ for each i. If possible, suppose that $E \subset K_i$ for some i, $1 \leq i \leq n$. Let $p, q \in E$, $p \neq q$. Since p, q are in E as well as in K_i, we have $p \overset{A}{-} q$ and $p \overset{B}{-} q$, which is a contradiction as $A + B$ separates points of X. Hence E must intersect at least two of the sets

$K_1, K_2, \ldots, K_n$. We call these sets S_E and T_E. If F is a nontrivial A-set such that $F \neq E$, $F \neq F_i$ $(1 \leq i \leq m)$, then the unordered pairs (S_E, T_E) and (S_F, T_F) have to be different. For if $(S_E, T_E) = (S_F, T_F)$ then $E \cap S_E$, $E \cap T_E$, $F \cap S_E$, $F \cap T_E$ are all nonempty and since $E \neq F$, $S_E \neq T_E$, we have a contradiction to Lemma 2. Thus $E \to (S_E, T_E)$ is a one-to-one map of the collection of nontrivial A-sets other than F_i's $(1 \leq i \leq m)$ to the set of unordered pairs of $\{K_1, K_2, \ldots, K_n\}$. Hence there are at most $m + {}^nC_2$ nontrivial A-sets. Similarly there are at most $n + {}^mC_2$ nontrivial B-sets.

Finally we get the following result from Theorems 8 and 4 and Corollary 5.

COROLLARY 9. *Let $C_R(X) = A \oplus B$. If X is a finite union of A-sets and B-sets, then all A-sets and all B-sets are retracts of X.*

REFERENCES

1. W. G. Bade, *The Banach space $C(S)$*, Lecture Notes 26, Aarhus Univ., Aarhus, 1971.
2. J. W. Baker, *Some uncomplemented subspaces of $C(X)$ of the type $C(Y)$*, Studia Math. **36** (1970), 85–103.
3. S. D. Fisher, *The decomposition of $C_r(K)$ into the direct sum of subalgebras*, J. Funct. Anal. **31** (1979), 218–223.
4. D. E. Marshall and A. G. O'Farrell, *Uniform approximation by real functions*, Fund. Math. **104** (1979), 203–211.
5. R. D. Mehta and M. H. Vasavada, *On the direct sum decomposition of an algebra of continuous functions*, Bull. Austral. Math. Soc. **32** (1985), 69–71.
6. ——, *Algebra direct sum decomposition of $C_R(X)$*, Proc. Amer. Math. Soc. **98** (1986), 71–74.
7. Z. Semadeni, *Banach spaces of continuous functions*, Vol. 1, Monografie Matematyczne, Warszawa, 1971.
8. H. Yoshizawa, *On simultaneous extension of continuous functions*, Proc. Imp. Acad. Tokyo **20** (1944), 653–654.

DEPARTMENT OF MATHEMATICS, SARDAR PATEL UNIVERSITY, VALLABH VIDYANAGAR-388 120, GUJARAT, INDIA