PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 1, May 1987

SIMILARITY OF A LINEAR STRICT SET-CONTRACTION AND THE RADIUS OF THE ESSENTIAL SPECTRUM

MAU-HSIANG SHIH

ABSTRACT. If A is a bounded linear operator on a Hilbert space, define $r_e(A)$, the essential spectral radius of A, by

$$r_e(A) := \sup\{|\lambda| \colon \lambda \in \operatorname{ess}(A) = \text{ essential spectrum of } A\}.$$

It is shown that

$$r_e(A) = \inf\{\alpha(S^{-1}AS) | S \colon H \to H \text{ is a bounded invertible linear map}\},$$

where α is the Kuratowski measure of noncompactness. As a consequence, a charcterization of the similarity of a linear strict set-contraction is obtained.

1. This paper applies a classical result of Rota with known results about Browder's essential spectrum to give a new formula for the radius of the essential spectrum of a bounded linear operator in Hilbert space.

Throughout the paper, H will denote a complex Hilbert space with norm $\|\cdot\|$. By an operator we always mean a bounded linear transformation on H. The identity operator is denoted by I. The spectrum of an operator A is denoted by $\sigma(A)$ and the spectral radius is denoted by $r(A)$. Recall that an operator A is called a *strict contraction* if $\|A\| < 1$ [**2**, p. 82]. The classical result of Rota's similarity theorem [**6; 2**, p. 81] asserts that an operator on H is similar to a strict contraction if and only if its spectrum is included in the interior of the unit disc. There is an elegant quantitative version of Rota's theorem [**2**, p. 77]; it asserts that the spectral radius of A is always equal to the infimum of norms of all conjugates (i.e., transformation by similarities) of A. Seeking to characterize the similarity of a linear *strict set-contraction* (the perturbation of a strict contraction by a compact operator is a strict set-contraction), we may therefore raise the question: What kind of spectrum included in the interior of unit disc has to be similar to a linear strict set-contraction? Thanks to the works of Kuratowski [**3**], Browder [**1**], Nussbaum [**5**], and Leggett [**4**], we are capable of solving this full problem. Our main results are proved in §3 and §2 contains some preliminary notions and lemmas.

The author wishes to thank the referee for his valuable suggestions.

2. We first recall the "measure of noncompactness," a notion which was introduced by Kuratowski in 1930 [**3**].

Let Ω be a nonempty subset of H. Kuratowski [**3**] defined the measure of noncompactness of Ω, in symbols $\alpha(\Omega)$, to be

$$\inf\{\varepsilon > 0 \colon \Omega \text{ can be covered by a finite number of sets of diameter less than or equal to } \varepsilon\}.$$

Received by the editors March 24, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A15, 47B37.
Key words and phrases. Essential spectrum, k-set-contraction, measure of noncompactness, similarity of operators.

It follows immediately that a subset Ω of H has a compact closure if and only if $\alpha(\Omega) = 0$. Closely associated with the notion of the measure of noncompactness is the concept of "k-set-contraction," defined as follows. Let T be a continuous map from Ω into H; then T is called a *k-set-contraction* if there exists a constant $k \geq 0$ such that for any nonempty bounded set $D \subset H$ we have $\alpha[T(D)] \leq k\alpha(D)$. If T is a k-set-contraction, Nussbaum [**5**] defined the *measure of noncompactness* of T, in symbols $\alpha(T)$, to be

$$\inf\{k > 0 \colon T \text{ is a } k\text{-set-contraction}\}.$$

If $\alpha(T) < 1$, then T is called a *strict set-contraction.* If $A := C + S$, where C is a compact operator and S a strict contraction, then A is a strict set-contraction.

We may recall another notion introduced by Browder [**1**], that of the essential spectrum of an operator. Browder defined the *essential spectrum* of an operator A, in symbols $\operatorname{ess}(A)$, to be the set of $\lambda \in \sigma(A)$ such that at least one of the following conditions holds:

(i) $R(\lambda I - A)$, the range of $\lambda I - A$, is not closed.

(ii) λ is a limit point of $\sigma(A)$.

(iii) $\bigcup_{\nu=1}^{\infty} N(\lambda I - A)^{\nu}$ is infinite dimensional, where $N(\lambda I - A)^{\nu}$ denotes the null space of $(\lambda I - A)^{\nu}$.

If A is a an operator on H, define $r_e(A)$, the essential spectral radius of A, by

$$r_e(A) := \sup\{|\lambda| \colon \lambda \in \operatorname{ess}(A)\}.$$

Nussbaum proved in [**5**] that

$$r_e(A) = \lim_{n\to\infty} (\alpha(A^n))^{1/n}.$$

Note that the above formula should be extended to read

$$r_e(A) = \inf_{n\geq 1} (\alpha(A^n))^{1/n} \leq \alpha(A).$$

For an operator A, we observe that A is a $\|A\|$-set-contraction and hence $\alpha(A) \leq \|A\|$.

The following lemmas will be needed in the proofs of our results. When we say an operataor is finite dimensional, we shall mean its range is finite dimensional.

LEMMA 1 (NUSSBAUM [**5**]). *Let A be an operator on H and $r > r_e(A)$. Then there exists a finite dimensional operator F on H such that $\sigma(A + F) \subset \{\lambda \in \mathbf{C} \colon |\lambda| \leq r\}$.*

LEMMA 2. *Similar operators on H have the same essential spectrum.*

PROOF. Let A be an operator on H and $B = P^{-1}AP$ for some invertible operator P on H. Then the conclusion follows from the following identities. (The identity (ii) below is a known result.)

(i) $P^{-1}[R(\lambda I - A)] = R(\lambda I - B)$, $P^{-1}[\operatorname{cl}(\lambda I - A)] = \operatorname{cl}(\lambda I - B)$, where $\operatorname{cl}(\lambda I - A)$ denotes the closure of $(\lambda I - A)(H)$.

(ii) $\sigma(A) = \sigma(B)$.

(iii) $P^{-1}[N(\lambda I - A)^{\nu}] = N(\lambda I - B)^{\nu}$ for each positive integer ν.

3. Our main result is the following:

THEOREM 1. *Let A be an operator on H. Then*

$$r_e(A) = \inf\{\alpha(S^{-1}AS) | S\colon H \to H \text{ is a bounded invertible linear map}\}.$$

PROOF. Lemma 2 implies that if $S\colon H \to H$ is bounded and invertible,

$$r_e(A) = r_e(S^{-1}AS) \leq \alpha(S^{-1}AS).$$

Thus one has

$$r_e(A) \leq \inf\{\alpha(S^{-1}AS) | S \text{ is one-one, onto, and linear}\}.$$

To prove the opposite inequality, take $\varepsilon > 0$ and use Lemma 1 to find a finite-dimensional operator F such that

$$r(A + F) \leq r_e(A) + \varepsilon/2.$$

By Rota's theorem, there exists S such that

$$\|S^{-1}(A + F)S\| \leq r(A + F) + \varepsilon/2 \leq r_e(A) + \varepsilon.$$

Finally we have (using basic properties of the seminorm α)

$$\|S^{-1}(A + F)S\| \geq \alpha(S^{-1}(A + F)S) = \alpha(S^{-1}AS + S^{-1}FS) = \alpha(S^{-1}AS).$$

Here we have used that $\|B\| \geq \alpha(B)$ and $\alpha(B + C) = \alpha(B)$ for any compact linear map C. This proves that

$$\alpha(S^{-1}AS) \leq r_e(A) + \varepsilon$$

and since $\varepsilon > 0$ was arbitrary, the proof is complete.

THEOREM 2. *Let A be an operator on H. Then A is similar to a linear strict set-contraction if and only if $r_e(A) < 1$.*

PROOF. This is immediate from Lemma 2 and Theorem 1 above.

REFERENCES

1. F. E. Browder, *On the spectral theory of elliptic differential operators*. I, Math. Ann. **142** (1961), 22–130.
2. P. R. Halmos, *A Hilbert space problem book*, 2nd ed., revised and enlarged, Springer-Verlag, New York, Heidelberg, and Berlin, 1982.
3. C. Kuratowski, *Sur les espaces complets*, Fund. Math. **15** (1930), 301–309.
4. R. Leggett, *Remarks on set-contractions and condensing maps*, Math. Z. **132** (1973), 361–366.
5. R. D. Nussbaum, *The radius of the essential spectrum*, Duke Math. J. **38** (1970), 473–478.
6. G.-C. Rota, *On models for linear operators*, Comm. Pure Appl. Math. **8** (1960), 469–472.

DEPARTMENT OF MATHEMATICS, CHUNG YUAN UNIVERSITY, CHUNG-LI, TAIWAN, REPUBLIC OF CHINA