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A SMALL ARITHMETIC HYPERBOLIC THREE-MANIFOLD

TED CHINBURG

ABSTRACT. The hyperbolic three-manifold which results from (5,1) Dehn surgery on the complement of a figure-eight knot in S^3 is arithmetic.

I. Introduction. Let M be the complete orientable hyperbolic three-manifold which results from (5,1) Dehn surgery on the complement of the figure-eight knot K in S^3. In this note we will prove

THEOREM 1. *M is arithmetic.*

A precise description of M as an arithmetic manifold is given in the summary at the end of this paper. One consequence of Theorem 1 and the results of Borel in [**1**] is that

$$\text{Volume}(M) = 12 \cdot 283^{3/2} \varsigma_k(2)(2\pi)^{-6},$$

where $\varsigma_k(s)$ denotes the Dedekind zeta function of the unique quartic field k of discriminant -283. Another consequence of Theorem 1 and Borel's work is that there exist infinitely many minimal elements in the set of manifolds commensurable to M.

By work of Jørgenson and Thurston (see [**7**, §6.6]), the set of volumes of complete orientable hyperbolic three-manifolds is a well-ordered subset of $\mathbf{R}$ of order type ω^ω. In particular, there is a minimal element v_1 in this set. In [**4**] Meyerhoff conjectured that M has volume v_1, but this is shown to be not true by Weeks [**9**]. Weeks proved that the manifold M' obtained by (5,1), (5,2) Dehn surgery on the complement of the Whitehead link in S^3 has

$$\text{Volume}(M') = 0.9427\ldots < \text{Volume}(M) = 0.9812\ldots.$$

The author and Jørgenson have proved that Weeks' manifold M' is also arithmetic (to appear), but the minimal volume v_1 remains unknown. For further discussion of the volumes of hyperbolic three-manifolds and orbifolds, see Thurston [**7**, **8**], Milnor [**5**], Borel [**1**], and Chinburg and Friedman [**2**].

II. Proof of Theorem 1.

LEMMA 1. *The fundamental group $\pi_1(M)$ is generated by two elements $\bar{\alpha}$ and $\bar{\beta}$, which are subject to the relations $f(\bar{\alpha}, \bar{\beta}) = g(\bar{\alpha}, \bar{\beta}) = 1$, where*

$$f(a,b) = (ab^{-1}a^{-1}b)a(ab^{-1}a^{-1}b)^{-1}b^{-1}, \tag{1}$$

$$g(a,b) = a^4b^{-1}aba^{-2}bab^{-1}a. \tag{2}$$

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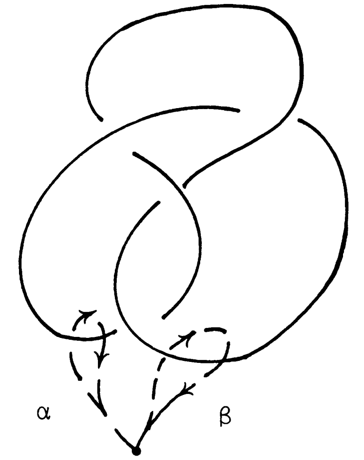

FIGURE 1

PROOF. Following Riley [6] and Milnor [5], we observe that $\pi_1(S^3 - K)$ is generated by the two loops α and β pictured in Figure 1, which are subject to the single relation $f(\alpha, \beta) = 1$.

We take α as a meridian for a torus neighborhood T of K in S^3. A positively oriented longitude l on T is given by $(\alpha^{-1}\beta^{-1}\alpha)\beta\alpha^{-1}(\alpha^{-1}\beta\alpha\beta^{-1}\alpha)$. By the definition of hyperbolic Dehn surgery (cf. Thurston [7]). $\pi_1(M)$ is isomorphic to the quotient of $\pi_1(S^3 - K)$ by the additional relation $\alpha^5 l = g(\alpha, \beta) = 1$. □

LEMMA 2. *There is a representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbf{C})$ with the following properties. The induced projective representation $\bar{\rho}\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$ is discrete and faithful, and M is isometric to $H^3/\bar{\rho}(\pi_1(M))$. Let $A = \rho(\bar{\alpha})$ and $B = \rho(\bar{\beta})$. There are nonzero $\lambda, \xi, r \in \mathbf{C}$ such that $|\lambda| \neq 1 \neq |\xi|$ and*

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} \xi & 0 \\ r & \xi^{-1} \end{bmatrix}. \tag{3}$$

PROOF. Since M is an orientable hyperbolic three-manifold, M is isometric to $H^3/\rho_1(\pi_1(M))$ for some discrete faithful representation $\rho_1\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbf{C})$. Let A and B in $\mathrm{SL}_2(\mathbf{C})$ have images $\overline{A} = \rho_1(\bar{\alpha})$ and $\overline{B} = \rho_1(\bar{\beta})$ in $\mathrm{PSL}_2(\mathbf{C})$. Let I be the 2×2 identity matrix. Then $f(A, B) = \pm I$ and $g(A, B) = \pm I$. By multiplying A by ± 1 and B by ± 1, we may make $f(A, B) = g(A, B) = I$, so that A and B give a representation $\rho_2\colon \pi_1(M) \to \mathrm{SL}_2(\mathbf{C})$ lifting ρ_1. Because M is compact (cf. Thurston [7]), A and B are hyperbolic.

Suppose that A and B have a common nonzero eigenvector. Then $(AB)^{-1}BA$ is unipotent. Since $\pi_1(M)$ can have no nontrivial parabolic elements, this would imply $(\bar{\alpha}\bar{\beta})^{-1}\bar{\beta}\bar{\alpha} = 1$, contradicting the fact that $\pi_1(M)$ is nonabelian. Hence A and B have no common nonzero eigenvector.

We may now find a basis $\{v_1, v_2\}$ for $\mathbf{C}^2$ such that $Av_1 = \lambda v_1$ and $Bv_2 = \xi^{-1}v_2$ for some nonzero $\lambda, \xi \in \mathbf{C}$ such that $|\lambda| \neq 1 \neq |\xi|$. Since v_2 is not an eigenvector of A, we may multiply v_1 by a nonzero scalar to have $Av_2 = \lambda^{-1}v_2 + v_1$. Relative to the basis $\{v_1, v_2\}$, A and B now have the form in (3) for some $r \in \mathbf{C}$, where $r \neq 0$ since

v_1 is not an eigenvector of B. The resulting representation $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbf{C})$ is conjugate to ρ_2 in $\mathrm{SL}_2(\mathbf{C})$, and thus has the properties required. □

LEMMA 3. *The numbers λ, ξ, and r of Lemma 2 are algebraic integers, and $\lambda = \xi$. The minimal polynomial of r is $Q(z) = z^4 - 8z^3 + 22z^2 - 25z + 11$. There is a root η of $P(z) = z^4 + z^3 - 1$ such that $r = 1 - \eta/(\eta^2 - 1)$ and $\xi^2 + \gamma\xi + 1 = 0$, where $\gamma = \eta^2 - 1$.*

PROOF. The equation $f(A, B) = I$ is equivalent to $DA = BD$, where $D = AB^{-1}A^{-1}B$. Compare the entries in the first row of DA to those in the first row of BD. Since these entries cannot both be zero, we find that one of the following is true:

$$\lambda = \xi^{-1} \quad \textit{and} \quad (\xi^{-2} - r)(\xi^2 + r) + r = 0, \quad \textit{or} \tag{4}$$

$$\lambda = \xi \quad \textit{and} \quad (r - 1)(2 - r - \xi^2 - \xi^{-2}) = 1. \tag{5}$$

Let $c_{i,j}$ be the (i, j) entry in $g(A, B) = I$. A computation shows that $c_{2,1} = 0$ and (4) imply that either $\xi = 1$ or $r = 0$. Both of these possibilities are excluded by Lemma 2, so (5) must be true. To arrive at a polynomial equation r must satisfy, one may write out the terms of $c_{1,1}^2 = 1$ and use (5) to eliminate the appearance of λ and ξ. One finds that r must be a root of $(z - 1)^7(z^2 - z + 1)Q(z)^2$, where $Q(z)$ is as in the statement of Lemma 3. Clearly (5) implies $r \neq 1$, while if $r^2 - r + 1 = 0$ then (5) implies $|\xi| = 1$, contradicting Lemma 2. Hence r is a root of $Q(z)$.

Elementary calculations show that $r = 1 - \eta/(\eta^2 - 1)$ for some root η of the irreducible polynomial $P(z) = z^4 + z^3 - 1$. From (5) we have $(\xi + \xi^{-1})^2 = 4 - r - 1/(r - 1) = \gamma^2$, where $\gamma = \eta^2 - 1$. Hence $\xi + \xi^{-1} = \pm\gamma$. Calculation shows $g(A, B) = -I$ if $\xi + \xi^{-1} = \gamma$, so we must have $\xi + \xi^{-1} = -\gamma$ and the lemma is proved. □

We will now prove Theorem 1.

With the notations of Lemmas 2 and 3, let $\mathbf{B}$ be the vector space over the field $k = \mathbf{Q}(\eta)$ with basis $\{I, A, B, AB\}$. We have $A^2 + \gamma A + I = B^2 + \gamma B + I = (AB)^2 + (\gamma^2 - 2 + r)AB + I = 0$. Since γ and r are in k, one checks that $\mathbf{B}$ is a quaternion algebra over k. There are two real places of k, and no nonzero solutions $(x, y, z) \in \mathbf{R}^3$ to the equation $\det(xI + yA + zB) = 0$ when A and B are embedded into $M_2(\mathbf{C})$ via an embedding over either real place of k. Hence $\mathbf{B}$ is ramified at the two real places of k.

Let F be the field $k(\xi)$, and let $\underline{0}_F$ be the integers of F. By Lemma 3, $\mathbf{B}$ splits over F. Let $\mathcal{D}$ be the maximal order $\mathbf{B} \cap M_2(\underline{0}_F)$ in $\mathbf{B}$, and let $\mathcal{D}^1 = \mathbf{B} \cap \mathrm{SL}_2(\underline{0}_F)$. The field k has a unique complex place ∞. The completion k_∞ of k at ∞ is isomorphic to $\mathbf{C}$, and we have an isomorphism $f_\infty\colon \mathbf{B} \otimes_k k_\infty \xrightarrow{\sim} M_2(\mathbf{C})$.

Define $\Gamma^1_{\mathcal{D}}$ to be the image of $f_\infty(\mathcal{D}^1) \subseteq \mathrm{SL}_2(\mathbf{C})$ in $\mathrm{PSL}_2(\mathbf{C})$. In [**1**] Borel proves that $\Gamma^1_{\mathcal{D}}$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$, and that

$$\mathrm{Volume}(H^3/\Gamma^1_{\mathcal{D}}) = \prod_{v \in R_f} (Nv - 1)|D_k|^{3/2}\zeta_k(2)(2\pi)^{-6}. \tag{6}$$

Here R_f denotes the set of finite places of k where $\mathbf{B}$ is ramified, and Nv is the norm of the finite place v. The discriminant D_k of k is -283, and $\zeta_k(s)$ denotes the Dedekind zeta function of k.

From Lemmas 2 and 3, we have $\bar{\rho}(\pi_1(M)) \subseteq \Gamma^1_{\mathcal{D}}$. We may now use the trivial estimate $\zeta_k(2) \geq 1$ in (6) and Meyerhoff's estimate Volume$(M) < 1$ in ([**4**], [**8**, p. 365]) to deduce that

$$1 \leq [\Gamma^1_{\mathcal{D}} \colon \bar{\rho}(\pi_1(M))] \leq \prod_{v \in R_f} (Nv - 1)^{-1} |D_k|^{-3/2} (2\pi)^6 \leq 13 \prod_{v \in R_f} (Nv-1)^{-1}. \tag{7}$$

A quaternion algebra over k is determined by the even number of places where it ramifies. In particular, there are an even number of places in R_f. The discriminant of the polynomial $P(z) = z^4 + z^3 - 1$ in Lemma 3 is -283, and $P(z)$ is irreducible modulo 2 and 3. Hence the rational primes 2 and 3 are inert in k. One may now check that $\# R_f$ even and $\prod_{v \in R_f}(Nv - 1) \leq 13$ imply that R_f is empty. The algebra $\mathbf{B}$ must therefore be isomorphic to $H_{\mathbf{Q}} \otimes_{\mathbf{Q}} k$, where $H_{\mathbf{Q}}$ denotes the Hamilton quaternion algebra over $\mathbf{Q}$.

Let v_{11} be the first-degree unramified place of k corresponding to the prime ideal $r\underline{0}_k$ lying over the rational prime 11. From the equation $\xi^2 + \gamma\xi + 1 = 0$ we find that v_{11} splits into two places v'_{11} and v''_{11} in $F = k(\xi)$. Let Γ_{11} be the group of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\underline{0}_F)$ which are in $\mathcal{D}^1$ and for which c is a nonunit at v'_{11}. Define $\Gamma_0(v'_{11})$ to be the image of $f_\infty(\Gamma_{11}) \subseteq \mathrm{SL}_2(\mathbf{C})$ in $\mathrm{PSL}_2(\mathbf{C})$.

Lemmas 2 and 3 show that $\bar{\rho}(\pi_1(M)) \subseteq \Gamma_0(v'_{11})$. Borel proves in [**1**, pp. 13–14] that $[\Gamma^1_{\mathcal{D}} : \Gamma_0(v'_{11})] = Nv_{11} + 1 = 12$. The fact, (7), and $R_f = \varnothing$ show that $\bar{\rho}(\pi_1(M)) = \Gamma_0(v'_{11})$. Since $H^3/\Gamma_0(v'_{11})$ is an arithmetic hyperbolic three-manifold, Theorem 1 now follows from Lemma 2. $\square$

The first statement in the following summary is shown by Godwin in [**3**].

SUMMARY. *The field $k = \mathbf{Q}(\eta)$ generated by a root of $\eta^4 + \eta^3 - 1 = 0$ is up to isomorphism the unique quartic field of discriminant -283. The quaternion algebra $\mathbf{B}$ over k is isomorphic to $H_{\mathbf{Q}} \otimes_{\mathbf{Q}} k$, where $H_{\mathbf{Q}}$ denotes the Hamilton quaternion algebra over $\mathbf{Q}$. Let v'_{11} be one of the two first-degree places over the rational prime 11 in the field $F = \mathbf{Q}(\xi)$ generated by a root of $\xi^2 + (\eta^2 - 1)\xi + 1 = 0$. The inclusion of k into F induces an injection $\mathbf{B} \to \mathbf{B} \otimes_k F \cong M_2(F)$. Let Γ_{11} be the group of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\underline{0}_F)$ which are in $\mathbf{B}$ and for which c is a nonunit at v'_{11}, where $\underline{0}_F$ denotes the integers of F. Let $\Gamma_0(v'_{11})$ be the image of Γ_{11} in $\mathrm{PSL}_2(\mathbf{C}) = \mathbf{B}^1_\infty/\{\pm I\}$ where $\mathbf{B}^1_\infty$ is the group of elements of reduced norm 1 in the completion $\mathbf{B}_\infty \cong M_2(\mathbf{C})$ of $\mathbf{B}$ at the unique complex place ∞ of k. Then (5,1) Dehn surgery on the complement of a figure-eight knot in S^3 yields a hyperbolic manifold isometric to $H^3/\Gamma_0(v'_{11})$.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19104

Current address: Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, California 94720