

BOOLEAN REDUCTS OF RELATION AND CYLINDRIC ALGEBRAS AND THE CUBE PROBLEM

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ABSTRACT. It is shown that not every Boolean algebra is the Boolean part of a nondiscrete relation or cylindric algebra, but every nonatomless Boolean algebra is. Solutions of Tarski's Cube Problem for nondiscrete relation and cylindric algebras are given.

Introduction. Relation algebras (RAs) and cylindric algebras (CAs) are Boolean algebras (BAs) endowed with additional structure. The question naturally arises: Which BAs are Boolean parts (i.e. Boolean reducts) of RAs or CAs?

This question can be considered as a representation problem for BAs, too: Which BA can be represented as a BA of some binary relations on a set U closed under composition, inversion, and containing the identity relation on U (this is the RA case); and which BA can be represented as a BA of n -ary relations closed under, roughly, first-order definability, i.e. containing with every relation R_1, \dots, R_k the relations first-order definable (with n variables) from them, too?

RA and CA_α denote the classes of RAs and CAs of dimension α respectively. There is a trivial way of turning every BA into a RA or a CA_α ; these RAs and CAs are called "Boolean" and "discrete" ones (cf. [7, p. 276 and 5, 1.3.11]). By [5, 1.2.14], every BA is the Boolean part of a nondiscrete CA_1 . J. D. Monk in 1981 and independently R. Laver in 1984 asked the question: Which BAs are Boolean parts (reducts) of nondiscrete CA_α s, $\alpha \geq 2$? Theorems 1 and 2 give partial answers.

To CA_α s there correspond another BA, the so-called zero-dimensional part $\mathfrak{B} \delta \mathfrak{A}$ of a $CA_\alpha \mathfrak{A}$. By [5, 2.4.35], every BA is the zero-dimensional part of a hereditarily nondiscrete CA_α (for all $\alpha > 0$). In the last part of the paper we use this fact (together with Ketonen's solution for BAs) to give a full solution of Tarski's Cube Problem for CA_α s as well as for RAs (of course we mean nondiscrete CA_α s and RAs here). See Theorems 3 and 4. This solves that part of Problem 2.4 of [5] which was still left open in [6, p. 127].

The definitions of RA and CA_α , both originating with Tarski, can be found e.g. in [5, 7, 8, 10] and in [5, 6] respectively. Below we recall that part of their definitions that will be needed in the present paper.

Received by the editors April 2, 1985 and, in revised form, August 19, 1985 and February 11, 1986.
1980 *Mathematics Subject Classification* (1985 Revision). Primary 03G05, 03G25; Secondary 03E15, 06E99.

LEMMA 2. *There is a BA \mathfrak{B} such that for any two disjoint nonzero elements $\underline{a}, \underline{b}$ of \mathfrak{B} there is a κ such that \underline{a} has a κ -partition while \underline{b} has no κ -partition.*

PROOF. We use the BA-construction (tree algebras) of [1, 2]. Let T be a tree satisfying (1) the height of T is ω and T has a single root and (2) $(\forall u, t \in T) [u \neq t \Rightarrow |A_u| \neq |A_t|$ and $|A_u|$ is regular and uncountable], where A_u is the set of immediate successors to u in T . Then the BA B_T generated by the principal filters of T satisfies the requirements for \mathfrak{B} in Lemma 2. This can be proved the same way as Claim 1 is proved in the proof of Theorem 2.4 of [1], but using a relativized version of Lemma 2.2 of [1]. Q.E.D.

We are ready to prove Theorem 1. Let $\mathfrak{B} \in \text{BA}$ be as stated in Lemma 2. Let $\mathfrak{A} \in \text{CA}_\alpha, \alpha \geq 2$, be nondiscrete. Then $D = d_{01} \cdot c_1 - d_{01} \neq 0, -d_{01} \neq 0$, and clearly $D \cdot -d_{01} = 0$. By Lemma 1 then \mathfrak{B} is not the Boolean reduct of \mathfrak{A} . By Theorem 5.3.17 and Definition 5.3.7 of [5], for every nondiscrete $\mathfrak{R} \in \text{RA}$ there is a nondiscrete $\mathfrak{C} \in \text{CA}_2$ such that the Boolean reducts of \mathfrak{R} and \mathfrak{C} coincide. Therefore \mathfrak{B} is not the Boolean reduct of any nondiscrete RA either. Q.E.D. (Theorem 1)

We note that one can easily state and prove Lemma 1 by writing $\mathfrak{A} \in \text{RA}, D = 1' \cdot [(-1)'; 1], -1'$, and $b_i = (a_i; 1) - 1'$ instead of $\mathfrak{A} \in \text{CA}, D = d_{01} \cdot c_1 - d_{01}, -d_{01}$, and $b_i = c_1 a_i - d_{01}$, respectively, this way obtaining a direct proof for RAs.

PROOF OF THEOREM 2(a). Let $\mathfrak{B} \in \text{BA}$ and $a \in \text{At}\mathfrak{B}$. For all $x, y \in B$ define $x \cup^d = x, 1' = a$, and

$$x; y = \begin{cases} 1^{\mathfrak{B}} & \text{if } 0^{\mathfrak{B}}, a \notin \{x, y\}, \text{ and } x \cdot y \neq 0, \\ 1^{\mathfrak{B}} - a & \text{if } 0^{\mathfrak{B}}, a \notin \{x, y\}, \text{ and } x \cdot y = 0, \\ 0^{\mathfrak{B}} & \text{if } 0^{\mathfrak{B}} \in \{x, y\}, \\ x & \text{if } a = y, \\ y & \text{if } a = x. \end{cases}$$

Let \mathfrak{R} be the algebra we obtain from \mathfrak{B} by endowing it with the above operations, i.e. $\mathfrak{R} = \langle \mathfrak{B}, ;, \cup^d, 1' \rangle$. Now it is not difficult to check that \mathfrak{R} is an integral RA. That \mathfrak{R} is isomorphic to a relation set algebra follows from results in [11]. We may assume that \mathfrak{B} is a Boolean set algebra with base U and that a is a one-element set. The algebra $\mathcal{E} = \mathcal{E}_{|U|}(\{1, 2, 3\})$ is defined in Definition 2.4 of [11] and shown to be an RA in [11, Theorem 2.5(5)]. That \mathcal{E} is an Rs follows from Theorem 5.19 of [11] since it is not difficult to check that \mathcal{E} satisfies condition 5.13(x). Finally, \mathfrak{R} is in Rs because it is a subalgebra of \mathcal{E} . Q.E.D. (Theorem 2(a))

PROOF OF THEOREM 2(b). Let $n \in \omega$. Assume that $\mathfrak{B} \in \text{BA}$ has at least $\pi n - 1$ atoms. Let $P = \{E: E \text{ is a partition of } n \text{ such that not every block of } E \text{ is one-element}\}$. Then $|P| = \pi n - 1$. Let $b: P \rightarrow \text{At}\mathfrak{B}$ be a one-one mapping. We shall write b_E instead of $b(E)$.

Case 1. $\sum(\text{Rg } b) = 1$ in \mathfrak{B} . Define $f(b_E) = \{s \in {}^n(n-1) : \ker(s) = E\}$ for $E \in P$ and $f(a) = \bigcup \{f(b_E) : b_E \leq a\}$ for $a \in B$. Then f is an isomorphism of \mathfrak{B} onto the Boolean part of the smallest Cs_n with base $n - 1$ (cf. e.g. [5, 2.4.69]).

Case 2. $\Sigma(\text{Rg } b) \neq 1$ in \mathfrak{B} . We may assume that \mathfrak{B} is a Boolean set algebra with base U . Let $K \stackrel{d}{=} U \sim \cup\{b_E : E \in P\}$. Then $K \neq 0$. We may assume that $|K| \geq \omega$. (There is a construction for finite K , too, but we want to avoid case-distinction.) We will construct $\langle H_\kappa : \kappa \in K \rangle$ with the following properties. Let $\bar{D} \stackrel{d}{=} \{s \in {}^n K : s \text{ is one-one}\}$.

$$(i) \cup\{H_\kappa : \kappa \in K\} = \bar{D}.$$

$$(ii) H_\kappa \cap H_\delta = 0 \text{ if } \kappa, \delta \in K, \kappa \neq \delta.$$

$$(iii) C_i H_\kappa = C_i \bar{D} \text{ for each } \kappa \in K \text{ and } i \in n.$$

Let $\langle K, +, - \rangle$ be any commutative group on K . Let $\langle K_\kappa : \kappa \in K \rangle$ be such that $\cup\{K_\kappa : \kappa \in K\} = K$, $(\forall \kappa \in K) |K_\kappa| \geq n$, and $(\forall \kappa, \delta \in K) [\kappa \neq \delta \Rightarrow K_\kappa \cap K_\delta = 0]$. Define

$$H_\kappa \stackrel{d}{=} \{s \in \bar{D} : \Sigma\{s_i : i \in n\} \in K_\kappa\}, \text{ for every } \kappa \in K.$$

Now $\langle H_\kappa : \kappa \in K \rangle$ clearly satisfies (i)–(ii). Let $\kappa \in K$, $i \in n$, and $s \in \bar{D}$ be fixed. Let $L \stackrel{d}{=} \{s_j : j \in n, j \neq i\}$, $\sigma \stackrel{d}{=} \Sigma L$, and $S \stackrel{d}{=} \{\delta - \sigma : \delta \in K_\kappa\}$. Then $|S| \geq n$, hence $S \sim L \neq 0$. Let $u \in S \sim L$. Let $z \in {}^n K$ be such that $z_i = u$ and $(\forall j \in n, j \neq i) z_j = s_j$. Then $z \in \bar{D}$ by $u \notin L$ and $\Sigma\{z_j : j \in n\} = \sigma + u \in K_\kappa$ by $u \in S$. Thus $z \in H_\kappa$, showing $C_i \bar{D} = C_i H_\kappa$. We have seen that (iii) holds for $\langle H_\kappa : \kappa \in K \rangle$. Define

$$f(b_E) \stackrel{d}{=} \{s \in {}^n K : \ker(s) = E\} \text{ for } E \in P$$

and

$$f(a) \stackrel{d}{=} \cup\{H_\kappa : \kappa \in a \cap K\} \cup \cup\{f(b_E) : b_E \leq a, E \in P\} \text{ for } a \in B.$$

Let \mathfrak{C} be the full Cs_n with base K . Then f is an embedding of \mathfrak{B} into the Boolean part $\mathfrak{B} \upharpoonright \mathfrak{C}$ of \mathfrak{C} by ${}^n K \sim \bar{D} = \cup\{f(b_E) : E \in P\}$, $f(1^\mathfrak{B}) = {}^n K$, and by (i)–(ii). Let $F \stackrel{d}{=} \{f(a) : a \in B\}$. We will show that F is closed under the operations of \mathfrak{C} . Let $i < j < n$. Then $d_{ij}^\mathfrak{C} = f(b_E)$ where E is the partition of n keeping i, j together and leaving all the other points separated. Let \mathfrak{M} be the smallest subalgebra of \mathfrak{C} . Then $M = \text{Sg}^{\mathfrak{B} \upharpoonright \mathfrak{C}}\{d_{ij}^\mathfrak{C} : i < j < n\}$ by [5, 2.2.24]. Hence $M \subseteq F$. Let $a \in B$ be arbitrary. Let $i \in n$. If $a \cap K = 0$ then $a = \Sigma\{b_E : b_E \leq a, E \in P\}$, hence $f(a) \in M$, therefore $c_i f(a) \in M \subseteq F$. Assume $a \cap K \neq 0$. Then $c_i f(a) = c_i \bar{D} + \Sigma\{c_i f(b_E) : b_E \leq a, E \in P\} \in M \subseteq F$ by (iii). We have seen that F is closed under c_i, d_{ij} ($i, j \in n$). Q.E.D. (Theorem 2(b))

REMARK. The proof of Theorem 2(b) goes through for Cs_α with $\alpha \geq \omega$. Then we obtain the following: If $\mathfrak{B} \in \text{BA}$ is such that $(\exists H \subseteq \text{At } \mathfrak{B})[|H| = |\alpha| \text{ and } (\forall L \subseteq H) \text{sup } L \text{ exists in } \mathfrak{B}]$, then \mathfrak{B} is the Boolean part of an integral Cs_α . The only change in the proof is that we use a Ws_α instead of a Cs_α in order to be able to speak about $\Sigma \text{Rg}(s)$ for $s \in 1^\mathfrak{C}$. \square

In the proofs of Theorems 3 and 4 we will use the notation of [5] without introducing it.

If U is a set, then PU denotes its powerset. The full relation set algebra with base U is the algebra

$$\langle P(U \times U), \cup, \cap, \sim, U \times U, \emptyset, |, ^{-1}, \text{Id}_U \rangle,$$

where $R|S$ denotes the composition of the relations R and S , R^{-1} is the inverse of the relation R , and Id_U denotes the identity relation $\{(u, u) : u \in U\}$ on U . In RA theory, the operations composition, inversion, and identity are denoted by $;$, $^{\cup}$, and $1'$ respectively. A relation set algebra (Rs) is a subalgebra of a full one. RA is a class of algebras defined with finitely many equations such that $\text{Rs} \subseteq \text{RA}$.

Let α be an ordinal. The full cylindric set algebra of dimension α with base U is the algebra

$$\langle P({}^{\alpha}U), \cup, \cap, \sim, {}^{\alpha}U, \emptyset, C_i, D_{ij} : i, j \in \alpha \rangle,$$

where if $X \subseteq {}^{\alpha}U$ and $i, j \in \alpha$ then

$$C_i X \stackrel{d}{=} \{s \in {}^{\alpha}U : (\exists z \in X)(\forall j \in \alpha, j \neq i)z_j = s_j\}$$

and

$$D_{ij} \stackrel{d}{=} \{s \in {}^{\alpha}U : s_i = s_j\}.$$

The unary operations C_i are called cylindrifications and correspond to quantification “ $\exists v_i$,” and the constants D_{ij} are called the diagonals and correspond to the equality “ $v_i = v_j$.” A cylindric set algebra of dimension α (Cs_{α}) is a subalgebra of a full one. Thus the universe of a Cs_{α} consists of α -ary relations and is closed under first-order (with α variables) definability. CA_{α} is a class of algebras defined by finitely many (if α is finite) equations such that $\text{Cs}_{\alpha} \subseteq \text{CA}_{\alpha}$.

In §1 we formulate the results and in §2 we give the proofs.

ACKNOWLEDGEMENTS. Theorems 1 and 2 of this paper were obtained in the stimulating atmosphere of the Boolean Algebra Meeting in Oberwolfach, Germany, January 1985. We want to express our thanks to G. Brenner, H. Dobbertin, and J. D. Monk for many stimulating discussions.

1. Formulating the results. We call an RA nondiscrete iff $1' \neq 1$ in it. Recall from [5] that a CA_{α} is nondiscrete iff $d_{01} \neq 1$ in it.

THEOREM 1. *There is a BA which is not the Boolean part of any nondiscrete CA_{α} , $\alpha \geq 2$; hence, it is also not the Boolean part of any nondiscrete RA.*

A relation algebra is called integral if $1'$ is an atom in it. Integral RAs have been extensively investigated, e.g. in [3, 7, 8, 10–12]. We call a CA_{α} \mathfrak{A} integral iff $|\text{Nr}_1 \mathfrak{A}| \leq 2$. (This expresses the fact that no proper subset of the base of \mathfrak{A} is definable.) This is equivalent to saying that d_{01} is an atom in $\text{Nr}_2 \mathfrak{A}$, or if $\alpha < \omega$ then to saying that $\prod\{d_{ij} : i < j < \alpha\}$ is an atom in \mathfrak{A} , or to saying that $\mathfrak{A}1 \mathfrak{A}$ is integral if $\alpha \geq 3$. Note that the only discrete integral CA_{α} is the two-element one.

THEOREM 2. (a) *Let $\mathfrak{B} \in \text{BA}$ have at least one atom. Then \mathfrak{B} is isomorphic to the Boolean reduct of an integral Rs (hence of a Cs_2 , too).*

(b) *Let $n \in \omega$ and let πn denote the number of partitions of n . Let $\mathfrak{B} \in \text{BA}$ have at least $\pi n - 1$ atoms. Then \mathfrak{B} is isomorphic to the Boolean reduct of some integral Cs_n .*

REMARK. Theorem 2(a) gives a complete characterization of the Boolean parts of each of the following classes: integral RAs, integral Rss, integral CA_{2s} , integral Cs_{2s} . In each case, the class of Boolean reducts is the class of all nonatomless BAs. The hypothesis of Theorem 2(b) is far from being necessary.

We note that for finite BAs the picture is complete: Let $\mathfrak{B} \in \text{BA}$ be finite with κ atoms, and let $n \in \omega$. Then \mathfrak{B} is the Boolean part of a hereditarily nondiscrete CA_n (Cs_n , integral Cs_n) iff $[\kappa \in \{\pi_\gamma n : \gamma \in n\} \text{ or } \kappa \geq \pi n]$, where $\pi_\gamma n$ denotes the number of partitions of n into at most γ classes. (This follows from [5, 2.4.69] and Theorem 2(b) herein.) Also, \mathfrak{B} is the Boolean part of a nondiscrete CA_n (Gs_n) iff $\kappa \geq 2^{(n-1)}$ (this can be proved from [5, 2.4.72]). Further, every finite BA is the Boolean reduct of an integral $\mathbb{R}s$ by Theorem 2(a); compare [10]. \square

THEOREM 3. *Let α be any ordinal. Then there is a $CA_\alpha \mathfrak{A}$ with cardinality $|\alpha \cup \omega|$ such that $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$ holds, $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{A}$ fails, and, in addition, the following hold.*

- (i) \mathfrak{A} is hereditarily nondiscrete if $\alpha > 0$.
- (ii) $\mathfrak{A} \in Gs_\alpha^{\text{reg}} \cap Lf_\alpha$ if $\alpha > 1$.
- (iii) $\mathfrak{A} \in ICs_\alpha \cap Lf_\alpha$ if $\alpha \geq \omega$.

THEOREM 4. *There is a denumerable nondiscrete RA \mathfrak{R} such that $\mathfrak{R} \cong \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R}$ but $\mathfrak{R} \not\cong \mathfrak{R} \times \mathfrak{R}$.*

2. Proofs.

PROOF OF THEOREM 1. We establish a property of Boolean reducts of nondiscrete CAs and then we construct a BA that does not have that property.

DEFINITION. An element of a BA has a κ -partition iff it is the supremum of an antichain of size κ . More formally: Let $x \in \mathfrak{B} \in \text{BA}$ and let κ be an ordinal. Then $\langle a_i : i \in \kappa \rangle \in {}^\kappa B$ is a κ -partition of x iff $(\forall i < j < \kappa) (a_i \neq 0 = a_i \cdot a_j \text{ and } x = \Sigma\{a_i : i \in \kappa\})$.

LEMMA 1. *Let $\mathfrak{A} \in CA_\alpha$, $\alpha \geq 2$. If $D =^d d_{01} \cdot c_1(-d_{01})$ has a κ -partition, then $-d_{01}$ has a κ -partition, too.*

PROOF. Let $\langle a_i : i \in \kappa \rangle$ be a κ -partition of D . Define $b_i =^d (c_1 a_i) - d_{01}$ for every $i \in \kappa$. We will show that $\langle b_i : i \in \kappa \rangle$ is a κ -partition of $-d_{01}$. $b_i \neq 0$ because of the following: Assume $b_i = c_1 a_i - d_{01} = 0$. By [5, 1.5.5] we have $c_1 a_i \cdot d_{01} = a_i$ since $a_i \leq d_{01}$. Now $c_1 a_i = (c_1 a_i \cdot d_{01}) + (c_1 a_i - d_{01}) = a_i$. But then $-a_i = c_1 - a_i \geq c_1 - d_{01} \geq a_i$, a contradiction. $b_i \cdot b_j = 0$ if $i \neq j$ because of the following:

$$c_1 a_i \cdot c_1 a_j = c_1(a_i \cdot c_1 a_j) = c_1(a_i \cdot d_{01} \cdot c_1 a_j) = c_1(a_i \cdot a_j) = c_1 0 = 0,$$

hence $b_i \cdot b_j = (c_1 a_i - d_{01}) \cdot (c_1 a_j - d_{01}) = 0$. $\Sigma\{b_i : i \in \kappa\} = -d_{01}$ because of the following: By [5, 1.2.6(i)] we have

$$\Sigma\{c_1 a_i : i \in \kappa\} = c_1 \Sigma\{a_i : i \in \kappa\} = c_1(d_{01} \cdot c_1 - d_{01}) = c_1 - d_{01}.$$

$(\Sigma\{x_i : i \in I\}) \cdot b = \Sigma\{x_i \cdot b : i \in I\}$ holds in every BA, hence

$$\Sigma\{c_1 a_i - d_{01} : i \in \kappa\} = (c_1 - d_{01}) - d_{01} = -d_{01}.$$

Q.E.D. (Lemma 1)

PROOF OF THEOREM 3. Let \mathfrak{B} be a denumerable BA such that $\mathfrak{B} \cong \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B}$ while $\mathfrak{B} \not\cong \mathfrak{B} \times \mathfrak{B}$. Such a BA exists by Ketonen's proof [9]. Let \mathfrak{C} be a minimal Cs_α , with infinite base if $\alpha \geq \omega$. Then \mathfrak{C} is directly indecomposable, $\mathfrak{C} \in Cs_\alpha^{reg}$, and $|C| \leq |\alpha \cup \omega|$. Construct the algebra $\mathfrak{A}' \in CA_\alpha$ as in the proof of 2.4.35 of [5]. Then $\mathfrak{A}' \subseteq {}^U\mathfrak{C}$ for some U by [5, statement (4), p. 316], hence $\mathfrak{A}' \in IGs_\alpha^{reg}$ and further $\mathfrak{A}' \in ICs_\alpha$ if $\alpha \geq \omega$ by $SP_\infty Cs_\alpha \subseteq I_\infty Cs_\alpha$ for $\alpha \geq \omega$ (cf. [5, 3.1.106]). $\mathfrak{A}' \in Lf_\alpha$ by [5, statement (5), p. 316]. \mathfrak{A}' is hereditarily nondiscrete for $\alpha > 0$ by (6) on p. 316 of [5]. Also, $|A'| = |\alpha \cup \omega|$ because $|C| \leq |\alpha \cup \omega|$, $|B| = \omega$ and since $A' \subseteq \{f \in {}^U C : \{f^{-1} * c : c \in C\} \text{ is a finite subset of } B\}$ by the definition of A' . It remains to show that $\mathfrak{A}' \cong \mathfrak{A}' \times \mathfrak{A}' \times \mathfrak{A}'$ and $\mathfrak{A}' \not\cong \mathfrak{A}' \times \mathfrak{A}'$. By $\mathfrak{B} \cong \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B}$ and by [5, 2.4.7], there are $b_0, b_1, b_2 \in B$ such that

$$\mathfrak{R}I_1 \mathfrak{B} = \mathfrak{B} \cong \mathfrak{R}I_{b_i} \mathfrak{B} \quad \text{for } i \in 3,$$

$$\sum \{b_i : i \in 3\} = 1 \quad \text{and} \quad b_i \cdot b_j = 0 \quad \text{for } i < j < 3.$$

The desired decomposition of \mathfrak{A}' now follows from [5, 2.4.35(ii)]. On the other hand, if $\mathfrak{A}' \cong \mathfrak{A}' \times \mathfrak{A}'$, then 2.4.3 and 2.4.35(i) of [5] yield $\mathfrak{B} \cong \mathfrak{B} \times \mathfrak{B}$ contradicting the choice of \mathfrak{B} . Q.E.D. (Theorem 3)

PROOF OF THEOREM 4. Let $\mathfrak{A} \in Gs_3$ be such that $\mathfrak{A} \cong \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$ but $\mathfrak{B} \not\cong \mathfrak{B} \times \mathfrak{B}$ for $\mathfrak{B} = {}^d \mathfrak{B} \delta \mathfrak{A}$. Such an \mathfrak{A} exists by inspecting the proof of Theorem 3. Let $\mathfrak{R} = \mathfrak{R} \alpha \mathfrak{A}$. Then $\mathfrak{R} \in RA$ by [5, 5.3.9]. By [5, 5.3.11] we have $\mathfrak{R} \alpha (\mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}) = \mathfrak{R} \alpha \mathfrak{A} \times \mathfrak{R} \alpha \mathfrak{A} \times \mathfrak{R} \alpha \mathfrak{A}$, thus $\mathfrak{R} \cong \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R}$. Assume $\mathfrak{R} \cong \mathfrak{R} \times \mathfrak{R}$. Let \mathfrak{B} denote the BA of all ideal elements of \mathfrak{R} , i.e. $\mathfrak{B} = \langle Ie \mathfrak{R}, +, \cdot, -, 0, 1 \rangle$, where $Ie \mathfrak{R} = {}^d \{x \in R : 1; x; 1 = x\}$. It can be checked that $Ie \mathfrak{R} = Zd \mathfrak{A}$, as follows. Let $x \in R = Nr_2 \mathfrak{A}$. Then $1; x = c_2(s_2^1 \cdot s_2^0 x) = c_2 c_0(d_{02} \cdot c_2 x) = c_0 c_2 x = c_0 x$ and similarly $x; 1 = c_1 x$. Thus $\mathfrak{B} = \mathfrak{B}$. Clearly, $Ie(\mathfrak{R} \times \mathfrak{R}) = Ie \mathfrak{R} \times Ie \mathfrak{R}$, hence $\mathfrak{R} \cong \mathfrak{R} \times \mathfrak{R}$ implies $\mathfrak{B} \cong \mathfrak{B} \times \mathfrak{B}$, contradicting $\mathfrak{B} \not\cong \mathfrak{B} \times \mathfrak{B}$. Q.E.D. (Theorem 4)

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