

THE FIRST EIGENVALUE OF A SCALENE TRIANGLE

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ABSTRACT. In this paper, we prove the lower bound

$$\lambda_1(T) \geq \frac{(L + \sqrt{4\pi A})^2}{16A^2}$$

for a triangle T with area A and perimeter L , where λ_1 is the first eigenvalue for the Laplace operator with Dirichlet boundary conditions. We also present analogous estimates for an arbitrary convex polygon.

Let T be a plane triangle with area A and perimeter L . In this note, we give an estimate for the first eigenvalue $\lambda_1(T)$ for the Laplace operator with Dirichlet boundary conditions. We prove

$$\text{THEOREM 1. } \lambda_1(T) \geq (L + \sqrt{4\pi A})^2 / 16A^2.$$

The idea of the proof is to use Cheeger's isoperimetric inequality [2] $\lambda_1 \geq \frac{1}{4}h^2$ where h is the infimum,

$$h = \inf_C \frac{\text{length}(C)}{\text{area}(\text{int}(C))};$$

here C runs over all closed curves in T . We then use geometric measure theory to find the curve which realizes this minimum. The inequality of Theorem 1 then follows from the sharp computation:

$$\text{THEOREM 2. } h(T) = (L + \sqrt{4\pi A}) / 2A.$$

REMARK. Theorem 2 is suggestive in that it expresses $h(T)$ as the average between L/A and the corresponding ratio for the circle with area A .

We then turn to the case of an arbitrary convex polygon P . To state our results succinctly, observe that there is a unique polygon P^* which has the same area as P , and whose angles are the same as those of P , but for which P^* has an inscribed circle—that is, there is a circle tangent to all the sides of P^* .

We then show

$$\text{THEOREM 3. } h(P) \geq h(P^*), \text{ with equality if and only if } P = P^*.$$

More precisely, if we denote by Θ_i half the angle at a vertex of P , we have

$$h(P) \geq \frac{\sqrt{\sum \cot(\Theta_i)} + \sqrt{\pi}}{\sqrt{A}},$$

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with equality if and only if P has an inscribed circle. Thus,

$$\lambda_1(P) \geq \frac{(\sqrt{\sum \cot(\Theta_i)} + \sqrt{\pi})^2}{4A}.$$

This suggests the conjecture that among all polygons with fixed area and angles, λ_1 is minimized for the unique polygon containing an inscribed circle.

An important step in the proof of Theorem 3 is the following

LEMMA 5 (§3). $L^2(P)/4A(P) \geq \sum \cot(\Theta_i)$, with equality if and only if P has an inscribed circle.

Noting that $\sum \cot(\Theta_i) > \pi$, this is a strengthening of the Euclidean isoperimetric inequality $L^2/4A \geq \pi$.

In §1, we prove the necessary facts from geometric measure theory. The proof of Theorem 2 is carried out in §2, and in §3 we prove Theorem 3.

1. Some geometric measure theory. Let V be a plane region with piecewise smooth boundary. We wish to estimate $h(V) = \inf_C(\text{length}(C)/\text{area}(\text{int}(C)))$, where C runs over closed curves in V .

It follows from general facts from geometric measure theory, and is proved, e.g. in [1] (see also [3]), that this infimum is realized by some curve D , which furthermore has mean curvature equal to h everywhere where it meets the interior of V . This last statement can be seen as follows: on any piece of D lying in the interior of V , deform D in a one-parameter family D_t along a vector field ξ supported near this piece.

Then

$$0 = \frac{d}{dt} \left(\frac{\text{length}(D_t)}{\text{area}(D_t)} \right) = \frac{(\text{length}(D_t))' - h \cdot (\text{area}(D_t))'}{\text{area}(D)}$$

On the one hand,

$$\frac{d}{dt}(\text{length } D_t) = \int_C \kappa \cdot \xi \, ds,$$

where $\kappa = \kappa\eta$ is the mean curvature vector, η is the normal. On the other hand,

$$\frac{d}{dt}(\text{area}(D_t)) = \int_C \eta \cdot \xi \, ds.$$

It follows that $\kappa \equiv h\eta$, as desired.

A segment of constant mean curvature κ in the plane is an arc of a circle of radius $1/\kappa$. It then follows that D consists of circular arcs of the same radius, together with pieces of the boundary of V .

We now investigate what happens at points where a circular arc of D meets the boundary of V . It is well known that at such points D remains C^1 , that is, that the circular arcs are tangent to the boundary. However, we have not been able to locate this fact in the literature, so we present our own proof, valid when V is a polygon.

LEMMA 1. *When V is a polygon, the circular arcs are tangent to the boundary of V .*

PROOF. Here and in what follows, we make use of the observation that if V' is a region contained in V but containing D , then D also realizes the isoperimetric

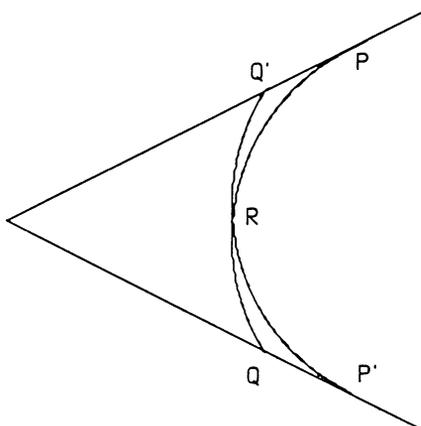


FIGURE 1

constant of V' . This is clear, because any curve contained in V' is also contained in V .

Suppose a circular arc from D meets a side at some nonzero angle. We then draw a line from this side which is tangent to the circular arc, so that we may assume without loss of generality that this circular arc is tangent to the adjacent side, as in Figure 1.

Denote by P and Q the points where this circular arc meets the sides, and reflect this picture in the angle bisector of the enclosed angle to obtain points P' and Q' . Let R denote the point where the circular arc meets its reflected image.

Denote by l_1 the length of the curve from Q to R , and l_2 the length of the curve from Q' to P to R . Let B denote the area of the region bounded by $\overline{Q'P}$, \overline{PR} , and $\overline{RQ'}$.

Replacing the curve \overline{RQ} by the curves $\overline{RP'}$ and $\overline{P'Q}$ gives a new curve D' satisfying

$$\frac{\text{length}(D')}{\text{area}(\text{int}(D'))} = \frac{\text{length}(D) + (l_2 - l_1)}{\text{area}(\text{int}(D)) + B} \geq h$$

from which it follows that $(l_2 - l_1)/B \geq h$. Similarly, replacing $\overline{PQ'}$ and \overline{PR} by $\overline{Q'R}$ shows that $(l_2 - l_1)/B \leq h$, hence $(l_2 - l_1)/B = h$.

It now follows that the isoperimetric ratio of D' equals that of D . But this is impossible, since D' is not a circular arc in a neighborhood of R . This contradiction establishes the lemma.

We now specialize to the case where V is a triangle T . We claim

LEMMA 2. D meets all three sides of T .

PROOF. Assume not. If D does not meet the side S of T , translate S parallel to itself to obtain a new triangle T' which has the same isoperimetric constant as T , but with the side corresponding to S tangent to D .

Now T' is similar to T , so we may enlarge T' by expanding it by some factor $\rho > 1$ to obtain T . But enlarging D by this same factor multiplies the isoperimetric ratio of D by $1/\rho$, contradicting the assertion that D realizes the isoperimetric constant of T .

2. A variational problem. We now consider a triangle T with area A and side lengths r_1, r_2, r_3 with $r_1 + r_2 + r_3 = L$. We wish to find the curve D minimizing the isoperimetric ratio.

From §1, we know that the minimum will occur when D consists of circular arcs, all of the same radius, meeting the sides tangentially. For each number a sufficiently small, let $D(a)$ denote the curve which consists of three circular arcs of radius a , cutting off the three corners of T , as in Figure 2. Let Θ_i denote the half-angles of the three corners.

LEMMA 3. *The length and area of $D(a)$ satisfy*

$$L(D(a)) = L + 2\pi a - 2a \left(\sum \cot(\Theta_i) \right),$$

$$A(D(a)) = A - a^2 \left(\sum \cot(\Theta_i) \right) + \pi a^2$$

so that

$$(*) \quad \frac{L(D(a))}{A(D(a))} = \frac{L + 2\pi a - 2a(\sum \cot(\Theta_i))}{A - a^2(\sum \cot(\Theta_i)) + \pi a^2}.$$

PROOF. We first observe that the length of the circular arc at the i th corner is $a(\pi - 2\Theta_i)$. Since $\sum 2\Theta_i = \pi$, this gives $2\pi a$ for the sum of the lengths of the circular arcs.

Furthermore, the length of one side from the vertex to the point of tangency of the circular arc is $a \cot(\Theta_i)$. This gives that $\text{length}(D(a)) = L + 2\pi a - 2a(\cot(\Theta_i))$. The expression $\text{area}(\text{int}(D(a))) = A - a^2(\sum \cot(\Theta_i)) + \pi a^2$ follows in a similar routine manner.

We now claim

LEMMA 4. $\sum \cot(\Theta_i) = L^2/4A$.

PROOF. We apply these considerations to the case where $D(a)$ is the circle inscribed in T . In this case, the sides of T joining a vertex of T to a point of tangency fill out all of T , so that $2a(\sum \cot(\Theta_i)) = L$. Similarly, we have $a^2(\sum \cot(\Theta_i)) = A$. Eliminating a from these two equations gives the lemma. Note also that we have found $a = 2A/L$ for the radius of the inscribed circle.

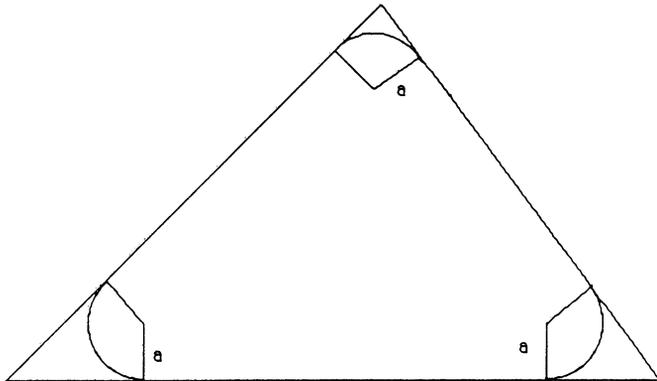


FIGURE 2

We now minimize (*) with respect to a , noting that for $a = 0$ and $a = 2A/L$, we get the same value L/A . From the usual isoperimetric inequality in the plane, we see that $L^2/4A > \pi$, so the minimum will occur for the value of a satisfying

$$(**) \quad a^2 \left(\frac{L^2}{4A} - \pi \right) - aL + A = 0$$

or

$$a = \frac{L \pm \sqrt{L^2 - 4A(L^2/4A - \pi)}}{2(L^2/4A - \pi)} = \frac{2A}{L \mp \sqrt{4\pi A}}.$$

The only meaningful value is the one lying between 0 and $2A/L$, so we find that $a = 2A/(L + \sqrt{4\pi A})$. For this value of a we see that

$$h(D(a)) = \frac{L + \sqrt{4\pi A}}{2A} = \frac{1}{a},$$

completing the proof of Theorem 2, and hence Theorem 1.

3. Convex polygons. In this section, we extend the results of the previous sections to arbitrary convex polygons. Here there are two difficulties not present in the previous sections. First of all, the minimizing curve need not touch all of the sides of the polygon. This can be seen by taking a triangle T , and chopping off corners outside the minimizing curve for T to obtain a polygon P with the same minimizing curve. Secondly, Lemma 4 is no longer valid unless P has an inscribed circle.

Turning to the second of these problems first, we will show

LEMMA 5. *For a convex polygon P ,*

$$\frac{L^2}{4A} \geq \sum \cot(\Theta_i)$$

with equality if and only if P has an inscribed circle.

REMARK. Note that Lemma 5 is a strengthening of the usual isoperimetric inequality for convex polygons. Indeed, if we denote half the exterior angle at the i th vertex by τ_i , we may rewrite Lemma 5 as $L^2/4A \geq \sum \tan(\tau_i)$. Using the fact that the $\tan(\tau_i) \geq \tau_i$, the right-hand side is then estimated by $\sum \tau_i = \pi$.

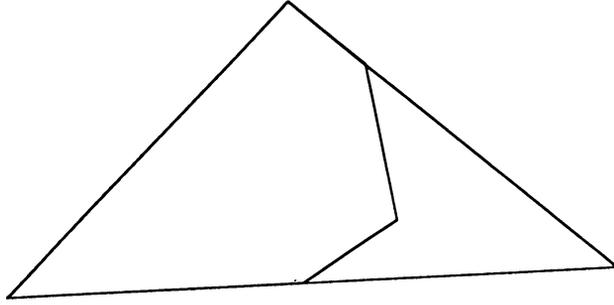
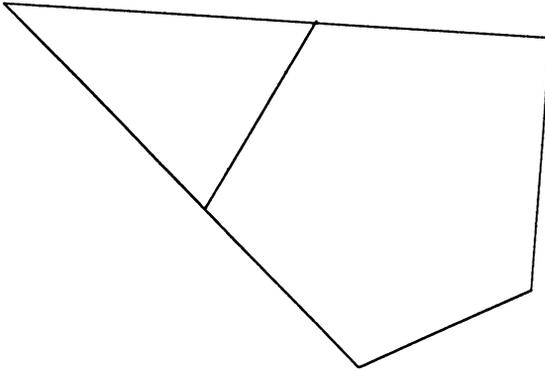
We will prove Lemma 5 by induction on the number n of sides of P . When $n = 3$, Lemma 5 becomes Lemma 4.

Consider polygons P' whose angles agree with the angles of P . It is easily seen that there is a polygon P^* , unique up to similarity, whose angles agree with those of P , and which has an inscribed circle. For this polygon, the proof of Lemma 4 shows that $L^2(P^*)/4A(P^*) = \sum \cot(\Theta_i)$.

Let us now parametrize the space of all polygons with these prescribed angles by the lengths of the sides. Let \mathcal{S} denote the space of all such polygons with area 1.

Note that if P has n sides, then we may prescribe at most the lengths of $n-2$ sides of P , keeping the angles fixed. It is easy to see from this that \mathcal{S} is homeomorphic to \mathbf{R}^{n-3} , and that L and A are continuous functions on \mathcal{S} .

We claim that L^2/A has a minimum on \mathcal{S} . To see this, let P_i be a sequence of polygons in \mathcal{S} with $L^2(P_i)/A(P_i)$ tending towards the minimum. Then, there will

FIGURE 3(a). $\Theta_j + \Theta_{j+1} < \pi/2$ FIGURE 3(b). $\Theta_j + \Theta_{j+1} > \pi/2$

be a convergent subsequence provided none of the side lengths get too short or too long. But if a side length gets too long, then $L^2(P_i)$ is large. Now suppose a side $S(i)$ is getting small, and let Θ_j and Θ_{j+1} be the two half-angles meeting $S(i)$. If $\Theta_j + \Theta_{j+1} < \pi/2$ then the polygon P_i is contained in a triangle with angles $2\Theta_j$ and $2\Theta_{j+1}$ and side $S(i)$. As $S(i)$ gets small, P_i cannot have area 1. If $\Theta_j + \Theta_{j+1} > \pi/2$, we may replace the side $S(i)$ by extending the sides meeting $S(i)$ to obtain a polygon $P^\#$ with fewer sides. As $S(i)$ gets sufficiently small, this changes $L^2/4A$ by an arbitrarily small amount. By induction, $L^2(P_i^\#)/4A(P_i^\#) \geq \sum \cot(\Theta_i^\#)$, and it is easily seen that $\sum \cot(\Theta_i^\#) > \sum \cot(\Theta_i)$, contradicting the assumption that $L^2(P_i)/4A(P_i)$ tends to the minimum, which is at most $\sum \cot(\Theta_i)$ by Lemma 4.

Finally, if $\Theta_j + \Theta_{j+1} = \pi/2$, then P_i is contained in a parallelogram with base length $S(i)$ and height of length $\leq L(P_i)$. It follows that as $S(i) \rightarrow 0$, the area of P_i cannot remain 1 without $L(P_i)$ getting large. See Figure 3 for an illustration of this argument.

Now let P be a polygon which minimizes L^2/A on \mathcal{S} , and let $P(a)$ be the polygon obtained from P as follows: The sides of $P(a)$ are parallel to the sides of P , and are a distance a from the sides of P and exterior to P . $P(a)$ will be similar to P if P has an inscribed circle, but not otherwise.

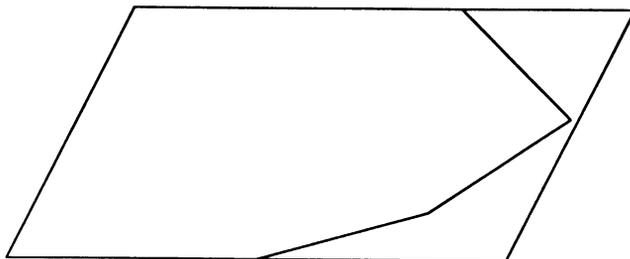


FIGURE 3(c). $\Theta_j + \Theta_{j+1} = \pi/2$

Letting $C = \sum \cot(\Theta_i)$, we calculate

$$L(P(a)) = L + 2aC,$$

$$A(P(a)) = A + a\left(\frac{1}{2}(L + L + 2aC)\right) = A + La + a^2C.$$

Differentiating with respect to a and setting equal to 0 gives $4AC = L^2$ showing that $L^2/4A$ takes the value C at the minimum. This gives the inequality in Lemma 5.

We now pick a side $S(i)$. Let us denote the length of $S(i)$ by L_i , and we let Θ_i and Θ_{i+1} be the half-angles meeting $S(i)$. Now let $P(b)$ be the polygon obtained from P by parallel translating the side $S(i)$ a distance b . We now calculate

$$L(P(b)) = L + a[\csc(\tau_i) - \cot(\tau_i) + \csc(\tau_{i+1}) - \cot(\tau_{i+1})],$$

$$A(P(b)) = A + a\left[\frac{1}{2}(L_i + (L_i - a \cot(\tau_i) - a \cot(\tau_{i+1})))\right]$$

$$= A + aL_i - a^2(\cot(\tau_i) + \cot(\tau_{i+1})),$$

where $\tau_i = \pi - 2\Theta_i$ is the i th exterior angle. Differentiating L^2/A with respect to b , and observing that $\csc(\tau_i) - \cot(\tau_i) = \cot(\Theta_i)$, we find that

$$2(\cot(\Theta_i) + \cot(\Theta_{i+1}))A = L \cdot L_i$$

so that

$$L_i = \frac{2(\cot(\Theta_i) + \cot(\Theta_{i+1}))A}{L}.$$

Since on S we have $A = 1$, and $L^2/4A = \sum \cot(\Theta_i)$ at a minimum value, this formula determines all the side lengths of a polygon at which $L^2/4A$ is a minimum. It follows that the minimum is achieved only at the polygon of S which has an inscribed circle. This establishes Lemma 5.

To prove Theorem 3, we first assume that the minimizing curve for P touches all the sides of P . In that case, formula (*) of Lemma 3 remains valid, so we may differentiate it to find the minimum value. We find that

$$(***) \quad h(P) = \frac{2(C - \pi)}{L - \sqrt{L^2 - 4AC} + 4\pi A},$$

where, as above, we have set $C = \sum \cot(\Theta_i)$.

We now claim that the expression on the right is $\geq (\sqrt{C} + \sqrt{\pi})/\sqrt{A}$. To see this, we divide both sides by $\sqrt{C} + \sqrt{\pi}$, and multiply by \sqrt{A} to obtain

$$2(\sqrt{C} - \sqrt{\pi})\sqrt{A} \geq L - \sqrt{L^2 - 4AC} + 4\pi$$

which further simplifies to

$$L - 2\sqrt{AC} + 2\sqrt{A\pi} \leq \sqrt{L^2 - 4AC + 4\pi A}.$$

Using $L^2/4A \geq C$ to check that both sides are positive and squaring yields $2A\sqrt{C}(\sqrt{C} - \sqrt{\pi}) \leq L\sqrt{A}(\sqrt{C} - \sqrt{\pi})$. The inequality then reduces to $L/2\sqrt{A} \geq \sqrt{C}$, establishing the claim. Note that when $L^2/4A = C$, which holds when $P = P^*$ has an inscribed circle by Lemma 4, the expression (***) simplifies to

$$\begin{aligned} h(P^*) &= \frac{2(C - \pi)}{L - \sqrt{L^2 - 4AC + 4\pi A}} = \frac{2(C - \pi)}{L - \sqrt{4\pi A}} \\ &= \frac{2(L^2 - 4\pi A)}{4A(L - \sqrt{4\pi A})} = \frac{L + \sqrt{4\pi A}}{2A} = \frac{\sqrt{C} + \sqrt{\pi}}{\sqrt{A}} \end{aligned}$$

and we get equality.

Now suppose the minimizing curve does not touch all the sides of P . Consider the new polygon P^\dagger obtained from P by parallel translating the sides until they touch the minimizing curve for P . Then $h(P) = h(P^\dagger)$, but $A(P^\dagger) < A(P)$, and the minimizing curve now touches all the sides of P^\dagger , so that the discussion above applies. We then have

$$h(P) = h(P^\dagger) \geq \frac{\sqrt{C} + \sqrt{\pi}}{\sqrt{A^\dagger}} \geq \frac{\sqrt{C} + \sqrt{\pi}}{\sqrt{A}} = h(P^*)$$

and the theorem is proved.

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