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NK_1 OF FINITE GROUPS

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ABSTRACT. For R any ring with unity, let $NK_1(R)$ denote the kernel of the homomorphism ε_*: $K_1(R[t]) \to K_1(R)$ induced by the augmentation ε: $t \to 0$. We show that if π is a finite group of square-free order, then $NK_1(Z\pi) = 0$.

For R a commutative ring with unity and π a finite group, let $G_0^R R\pi$ denote the Grothendieck group of the category of finitely generated, R-projective, left $R\pi$-modules. There is a $G_0^R R\pi$-module structure on $K_1 R\pi$: for $[M] \in G_0^R R\pi$ and $[N, \alpha] \in K_1 R\pi$, $[M] \circ [N, \alpha] = [M \overline{\otimes}_R N, 1_M \otimes \alpha]$ where the bar indicates the diagonal action of π on $M \otimes N$. See Bass [1] for his construction of $K_1 R\pi$ used above.

THEOREM 1. *Let A, B be commutative rings with unity, f: $A \to B$ a ring homomoprhism and π a finte group. Then the induced homomorphism f_*: $K_1 A\pi \to K_1 B\pi$ is $G_0^A A\pi$-linear.*

PROOF. For $[M] \in G_0^A A\pi$, $[N, \alpha] \in K_1 A\pi$,

$$f_*([M] \circ [N, \alpha]) = f_*\left(\left[M \overline{\otimes}_A N, 1_M \otimes \alpha\right]\right) = \left[B\pi \otimes_{A\pi} \left(M \overline{\otimes}_A N\right), 1_{B\pi} \otimes (1_M \otimes \alpha)\right],$$

$$\begin{aligned}[M] \circ f_*([N, \alpha]) &= \left[B \overline{\otimes}_A M\right] \circ \left[B\pi \otimes_{A\pi} N, 1_{B\pi} \otimes \alpha\right] \\ &= \left[\left(B \overline{\otimes}_A M\right) \overline{\otimes}_B \left(B\pi \otimes_{A\pi} N\right), (1_B \otimes 1_M) \otimes (1_{B\pi} \otimes \alpha)\right].\end{aligned}$$

Define

$$\phi\colon B\pi \otimes_{A\pi} \left(M \overline{\otimes}_A N\right) \to \left(B \overline{\otimes}_A M\right) \overline{\otimes}_B \left(B\pi \otimes_{A\pi} N\right)$$

by $\phi(bg \otimes (m \otimes n)) = (b \otimes gm) \otimes (1 \otimes gn)$ where $bg \in B\pi$, $m \in M$, $n \in N$. One can verify that ϕ is a $B\pi$-linear isomorphism with inverse ψ defined by

$$\psi((b \otimes m) \otimes (rg \otimes n)) = rbg \otimes \left(g^{-1}m \otimes n\right)$$

for $b \in B$, $rg \in B\pi$, $m \in M$, $n \in N$. It follows that $f_*([M] \circ [N, \alpha]) = [M] \circ f_*([N, \alpha])$ in $K_1 B\pi$.

COROLLARY 2. *For a homomorphism f: $A \to B$ of commutative rings, the induced homomorphism on functors f_*: $K_1 A[-] \to K_1 B[-]$ is a morphism of Frobenius modules over the Frobenius functor $G_0^A A[-]$.*

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COROLLARY 3. *$NK_1Z[-]$ is a $G_0Z[-]$-module.*

PROOF. The category of $G_0^AA[-]$-modules is an abelian category [4, p. 108], so $\ker f_*$ is a $G_0^AA[-]$-module. The ring homomorphism $Z \to A$ induces a morphism $G_0Z[-] \cong G_0^ZZ[-] \to G_0^AA[-]$, making $\ker f_*$ a $G_0Z[-]$-module. Now apply Corollary 2 with $A = Z[t]$, $B = Z$, and $f =$ augmentation.

COROLLARY 4. *For any finite group π, $NK_1Z\pi = \sum i_*(NK_1Z\rho)$ where the sum is taken over the set of all hyperelementary subgroups ρ of π, and i_* is the homomorphism induced by the inclusion $i\colon \rho \to \pi$.*

PROOF. It follows by the work of Lam [4, p. 123].

THEOREM 5. *If π is a finite group of square-free order, then $NK_1Z\pi = 0$.*

PROOF. By Corollary 4, we may assume that π is hyperelementary. Let $\pi = C_m \rtimes B$ where $C_m =$ cyclic group of order m, $|B| = p$ with p a prime, $p \nmid m$. Let $C_m = \langle a \rangle$, $B = \langle b \rangle$. Then π has the presentation $\pi = \langle a, b\colon a^m, b^p, bab^{-1}a^{-\alpha} \rangle$ where $\alpha^p \equiv 1 \pmod m$. If $\alpha \equiv 1 \pmod m$, then π is abelian and thus $NK_1Z\pi = 0$ [6, Theorem 2.2].

For a divisor d of m, let ζ_d denote a complex primitive dth root of unity. The twisted group algebra $Q(\zeta_d) \circ B$ has the additive structure of a free right $Q(\zeta_d)$-module based on B, and multiplicative structure determined by that in $Q(\zeta_d)$, by that in B, and the rule $b\zeta_d = \zeta_d^\alpha b$.

For any collection M of positive divisors of m, let $\mathcal{O}(M)$ denote the image of $Z\pi$ under the composite

$$Z\pi \to Q\pi \xrightarrow{\cong} \bigoplus_{d \mid m} Q(\zeta_d) \circ B \to \bigoplus_{d \in M} Q(\zeta_d) \circ B.$$

For each $d \in M$, there is a Cartesian square

$$\begin{array}{ccc} \mathcal{O}(M) & \xrightarrow{\phi} & \mathcal{O}(M - \{d\}) \\ \psi \downarrow & & \downarrow \\ \mathcal{O}(d) & \to & \mathcal{O}(d)/J \end{array}$$

where ϕ, ψ are projections and $J = \psi(\ker \phi)$ is the ideal of $\mathcal{O}(d)$ generated by $\prod_{e \in M + \{d\}} \phi_e(\zeta_d)$, and $\phi_e(x)$ is the minimal polynomial of ζ_e over Q [5, pp. 403–440]. Then $\phi_e(\zeta_d)$ is a unit of $Z[\zeta_d]$ if neither d/e nor e/d is a power of prime, and is associate in $Z[\zeta_d]$ to a prime q if $e/d = q^r$ for some $r > 0$ [5, Lemma 9.3]. Choosing d minimal in M and using the fact that m is square-free, we obtain

$$\prod_{e \in M - \{d\}} \phi_e(\zeta_d) \sim \prod_{\substack{p_i \text{ prime} \\ p_i d \in M}} \phi_{p_i d}(\zeta_d) \sim \prod_{\substack{p_i \text{ prime} \\ p_i d \in M}} p_i = \gamma$$

where " $\sim$ " means associate to. So for a minimal divisor $d \in M$, J is the ideal generated by γ.

If d is such that $B \to \operatorname{Aut}(Q(\zeta_d))$ has a nontrivial kernel, then $\mathcal{O}(d)$ is in fact that group ring $Z[\zeta_d]B$ and $NK_1Z[\zeta_d]B = 0$ [6, Theorem 2.2]. This occurs exactly when $d \mid (m, \alpha - 1)$.

Otherwise, B acts faithfully on $Q(\zeta_d)$. Let $F = Q(\zeta_d)^B$, the subfield of $Q(\zeta_d)$ left fixed by the action of B. The cyclic algebra $Q(\zeta_d) \circ B = (Q(\zeta_d)/F, b, 1)$ [**8**, p. 259] is a crossed-product algebra with trivial factor set and hence isomorphic to $M_p(F)$ [**8**, Corollary 29.8]. Then $\dim_F Q(\zeta_d) = p$ and by choosing the integral basis $\{1, \zeta_d, \ldots, \zeta_d^{p-1}\}$ for $Q(\zeta_d)/F$, the isomorphism $Q(\zeta_d) \circ B \to M_p(F)$ above restricts to $Z[\zeta_d] \circ B = \mathcal{O}(d) \to M_p(R)$ where R is the ring of integers in F.

For a nonzero prime $p \triangleleft R$, $P \cdot Z[\zeta_d] = P_1^e \cdots P_g^e$ where $P_1, \ldots, P_g$ are distinct maximal ideals in the Dedekind domain $Z[\zeta_d]$ and e is the ramification index. Since $e \mid p$, $e = 1$ or $e = p$. If $e = 1$ then P is unramified in $Z[\zeta_d]$. Otherwise $e = p$; but the only primes $P \triangleleft R$ that ramify are those for which $q \mid d$ where $Z \cap P = qZ$, and $p \nmid d$ since $p \nmid m$. Hence $e \neq 0$ in Z/qZ and P is tamely ramified in $Z[\zeta_d]$. So $Z[\zeta_d] \circ B$ is hereditary [**8**, Theorem 40.15] and hence regular. Thus $NK_i Z[\zeta_d] \circ B = 0$ for $i = 0, 1$ [**3**], and $i = 2$ [**7**].

$M_p(R)$ is a maximal order containing $\mathcal{O}(d)$ and $d \cdot M_p(R) \subset \mathcal{O}(d)$, so $\mathcal{O}(d)/(\gamma) \simeq M_p(R/(\gamma))$ [**5**, Proposition 10.2]. For each pair of primes p_i, p_j in the factorization of γ, $p_i Z + p_j Z = Z$ and so $p_i R = p_j R = R$. By the Chinese Remainder Theorem, $R/(\gamma) \simeq \oplus_i R/(p_i)$ and thus $M_p(R/(\gamma)) \simeq \oplus_i M_p(R/(p_i))$. Since $p_i \nmid d$, (p_i) is unramified in $Z[\zeta_d]$, hence in R as well. So $(p_i) = \prod_j P_{ij}$ where $\{P_{ij}\}$ are distinct maximal ideals in R. Thus $\oplus_i M_p(R/(p_i)) \cong \oplus_{i,j} M_p(R/P_{ij})$, a direct sum of matrix rings over fields, hence regular. So $NK_i(\mathcal{O}(d)/(\gamma)) = 0$ for $i = 0, 1, 2$.

The Mayer-Vietoris exact sequence resulting from the Cartesian square is

$$NK_2\mathcal{O}(d)/(\gamma) \to NK_1\mathcal{O}(M) \to NK_1\mathcal{O}(M - \{d\}) \oplus NK_1\mathcal{O}(d) \to NK_1\mathcal{O}(d)/(\gamma)$$

and thus $NK_1\mathcal{O}(M) \cong NK_1\mathcal{O}(M - \{d\})$.

By iterating this procedure, starting with M the set of all divisors of m and peeling off the minimal divisors $d \in M$, we obtain $NK_1 Z\pi \cong NK_1\mathcal{O}(M) = 0$, thus proving the theorem.

Higher N's are defined recursively by $N^{j+1}K_1 = N(N^j K_1)$ for $j = 1, 2, \ldots$. Using Corollary 2 with $A = Z[s, t]$, $B = Z[t]$ and, f: $s \to 0$ we can again use hyperelementary induction to compute $N^2K_1 Z\pi$. For π a hyperelementary group of square-free order, we tensor the Cartesian squares in the proof of Theorem 5 with $Z[t]$, producing new Cartesian squares and preserving regularity [**2**, Theorem 9.5]. Thus $NK_1 Z\pi[t] = 0$, and continuing inductively we obtain

COROLLARY 6. *$N^jK_1 Z\pi = 0$ for π a finite group of square-free order*, $j = 1, 2, \ldots$.

It may be worth noting that all the above results concerning NK_1 hold for NK_0 as well.

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