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A NONARCHIMEDEAN STONE-BANACH THEOREM

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ABSTRACT. If the spaces $C(T,R)$ and $C(S,R)$ of continuous functions on S and T are linearly isometric, then T and S are homeomorphic. By the classical Stone-Banach theorem the only linear isometries of $C(T,R)$ onto $C(S,R)$ are of the form $x \to a(x \circ h)$, where h is a homeomorphism of S onto T and $a \in C(S,F)$ is of magnitude 1 for all s in S. What happens if R is replaced by a field with a valuation? In brief, the result fails. We discuss "how" by way of developing a necessary and sufficient condition for the theorem to hold, along with some examples to illustrate the point.

Let K denote the real numbers R or the complex numbers C, let S and T be compact Hausdorff spaces, and let $C(T,K)$ and $C(S,K)$ denote the vector spaces of continuous maps of T and S into K, respectively, endowed with their supremum norms. If $C(S,K)$ and $C(T,K)$ are linearly isometric—under a map A, say—then the Stone-Banach theorem asserts the existence of a homeomorphism h of S onto T and a continuous function a mapping S into K, $|a(s)| \equiv 1$, such that for any x in $C(T,K)$ and any s in S, $(Ax)(s) = a(s)x(h(s))$. In other words, if a linear isometry exists between $C(T,K)$ and $C(S,K)$, then it must be of a very specific type, essentially just a change of variables.

In this paper we investigate what happens when S and T are compact 0-dimensional Hausdorff spaces and K is replaced by a nonarchimedean nontrivially valued field F. We show that in this setting there can be linear isometries other than the type mentioned above. A necessary and sufficient condition for a linear isometry A to be of the type mentioned above, what we call of "Stone-Banach" type, is that A map functions with disjoint cozero sets into functions with disjoint cozero sets.

NOMENCLATURE. Clopen means closed and open. S and T denote compact 0-dimensional Hausdorff spaces. F is a nontrivial nonarchimedean valued field and $C(S,F)$ and $C(T,F)$ denote the linear spaces of continuous maps of S and T into F, respectively, each with the supremum norm. $C(T,F)'$ and $C(S,F)'$ are the normed duals of $C(T,F)$ and $C(S,F)$, respectively. For each t in T, $t' \in C(T,F)'$ denotes the evaluation map at t. The analogous convention holds for points s in S. If U is a subset of S or T, k_U denotes the F-valued characteristic function of U.

1. Principal results.

DEFINITION 1. *Cozero sets.* For $x \in C(T,F)$, we define the cozero set of x to be $c(x) = \{t \in T \colon x(t) \neq 0\}$. A map B of $C(T,F)$ into $C(S,F)$ has the *disjoint cozero set property* if $c(x) \cap c(y) = \varnothing$ implies $c(Bx) \cap c(By) = \varnothing$ for $x, y \in C(T,F)$.

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LEMMA 1. *The linear span of the set of F-valued characteristic functions of clopen sets is dense in $C(T,F)$. Moreover, given any $x \in C(T,F)$ and $r > 0$, there exist disjoint clopen sets $U_1, \ldots, U_n$ and scalars $a_1, \ldots, a_n$ such that $\|x - \sum_{i=1}^n a_i k_{U_i}\| < r$.*

PROOF. We prove only the second, stronger assertion. Given $x \in C(T,F)$ and $r > 0$, for any t in T, there exists a clopen neighborhood $V(t,r)$ of t such that $|x(s) - x(t)| < r$ for any s in $V(t,r)$. As T is compact, a finite number of these, $V(t_i, r)$, $i = 1, \ldots, n$, say, cover T. By rewriting $\bigcup_i V(t_i, r)$ as a disjoint union $\bigcup_i U(t_i, r)$ in the standard way, we obtain a partition $\{U(t_i,r) : i = 1, \ldots, n\}$ of clopen subsets of T. Now $y = \sum_{i=1} x(t_i) k_{U(t_i,r)}$ is a uniform r-approximation of x; in other words $\|x - y\| < r$ and the proof is complete.

REMARK. As T is ultranormal (disjoint closed subsets may be separated by clopen sets), it is straightforward to verify that the weak topology $\sigma(C(T,F)', C(T,F))$ restricted to the set $T' = \{t' : t \in T\}$ of evaluation maps is such that T' is homeomorphic to T in its original topology under the map $t' \to t$.

DEFINITION 2. *Stone-Banach maps.* A linear isometry of $C(T,F)$ onto $C(S,F)$ is a *Stone-Banach map* if there exists a homeomorphism h of S onto T and an $a \in C(S,F)$, $|a(s)| \equiv 1$, such that for each x in $C(T,F)$ and s in S, $(Ax)(s) = a(s)x(h(s))$.

LEMMA 2. *A nontrivial continuous linear form f on $C(T,F)$ is a scalar multiple of an evaluation map if and only if $f(k_U) = 0$ or $f(k_V) = 0$ for every pair of disjoint, clopen sets U and V of T.*

PROOF. The necessity of the condition is evident. To prove sufficiency, let L denote the collection of clopen subsets U of T such that $f(k_U) \neq 0$. L has the following properties whose proofs we omit:

(a) L is closed with respect to the formation of finite intersections.

(b) Clopen supersets W of sets U in L also belong to L.

(c) A clopen subset W of T belongs to L if and only if W contains the intersection B of the sets in L.

(d) Since T is compact, B is not empty; since T is a 0-dimensional Hausdorff space, B is a singleton, say $\{t\}$.

(e) For some a in F, $f = at'$, where t is as in (d), on the linear span of the characteristic functions of clopen sets.

The desired result now follows from Lemma 1.

THEOREM. *For a linear isometry A mapping $C(T,F)$ onto $C(S,F)$, the following statements are equivalent:*

(a) *A has the disjoint cozero set property.*

(b) *Its adjoint A' maps each norm-one multiple of an evaluation map into a norm-one multiple of an evaluation map.*

(c) *A is a Stone-Banach map.*

PROOF. To see that (a) implies (b), let s' be an evaluation map on $C(S,F)$ and let U and V be disjoint, clopen subsets of T. Then, since A has the disjoint cozero set property, Ak_U and Ak_V are nonzero on disjoint subsets of S. Consequently s belongs to the cozero set of at most one of Ak_U, Ak_V, so $A's'$ satisfies the condition of Lemma 2; thus, for some scalar a and some point t of T, $A's' = at'$. Since A, and therefore A' are of norm one, it follows that a is of magnitude one.

Now assume that (b) holds, let $s \in S$, and let $A's' = at'$ ($|a| = 1$, $t \in T$). Define $h(s) = t$ and $a(s) = a$. Thus there are mappings $a\colon S \to F$ and $h\colon S \to T$ such that

(1) $$|a(s)| = 1 \quad \text{and} \quad A's' = a(s)h(s)' \quad \text{for each } s \text{ in } S,$$

and for each x in $C(T,F)$

(2) $$(Ax)(s) = s'(Ax) = (A's')x = a(s)h(s)'x = a(s)x(h(s)) \quad \text{for each } s \text{ in } S.$$

It remains to be shown that a is continuous and that h maps S homeomorphically onto T. Since S is compact and T is Hausdorff, we need only prove that h is a continuous bijection to prove the latter assertion.

Let $w \in C(T,F)$ be the constant with value 1. By (2), $a(s) = a(s)w(h(s)) = (Aw)(s)$ for all s in S; therefore $a = Aw$ is continuous.

To see that h is 1-1, let u and v be distinct points of S for which $h(u) = h(v)$. Choose $y \in C(S,F)$ such that $y(u) = 0$ and $y(v) = 1$. Now choose $x \in C(T,F)$ such that $y = Ax$. Let a and b be scalars of magnitude 1 such that $A'u' = ah(u)'$ and $A'v' = bh(v)'$. Since $h(u) = h(v)$, using the bracket notation $\langle\ ,\ \rangle$ for linear functionals,

$$\langle x, (1/a)A'u'\rangle = \langle x, (1/b)A'v'\rangle \quad \text{or} \quad (1/a)\langle Ax, u'\rangle = (1/b)\langle Ax, b'\rangle.$$

Thus $(1/a)\langle y, u'\rangle = (1/b)\langle y, v'\rangle$, so $(1/a)y(u) = 0 = (1/b)y(v) = 1/b$ which is contradictory. It follows that h is injective.

In order to show h to be continuous, recall that the continuity of a linear map A implies the weak continuity (i.e., continuity when domain and range carry their weak topologies) of A which implies the weak continuity of A'. As a and the mapping $s \to s'$ from S to $C(S,F)'$ in its weak-* topology are continuous and $|a(s)| \equiv 1$, the map $s \to (1/a(s))A's'$ is a continuous map from S to $C(T,F)'$ in its weak-* topology. By (1), its range is contained in T'. By restricting the codomain of this map to T', h is seen to be the composite of this map and the homeomorphism $t' \to t$ from T' to T. Therefore h is continuous.

Since h is continuous and S is compact, $h(S)$ is compact and therefore closed in T. If $h(S) \neq T$, choose a point t not in $h(S)$ and a clopen neighborhood U of t which does not meet $h(S)$. Then for any s in S

$$0 = \langle k_U, h(s)'\rangle = \langle k_U, a(s)(h(s))'\rangle = \langle k_U, A's'\rangle = \langle Ak_U, s'\rangle = (Ak_U)(s).$$

In other words, $Ak_U = 0$ even though $k_U \neq 0$ which contradicts the fact that A is an isometry. It follows that h is surjective and therefore that h maps S homeomorphically onto T.

Finally, to see that (c) implies (a), suppose that A is a Stone-Banach map. Let x and y be elements of $C(T,F)$ which have disjoint cozero sets. As $(Ax)(s) = a(s)x(h(s))$ and $(Ay)(s) = a(s)y(h(s))$, $s \in S$, Ax and Ay are seen to have disjoint cozero sets; in other words, A has the disjoint cozero set property and the proof is complete.

2. Examples. The following example produces a class of linear isometries which are not Stone-Banach maps.

Non-Stone-Banach maps. Let T be a compact Haudsorff 0-dimensional space with disjoint nonempty clopen subsets U and V such that U and V are homeomorphic in their relative topologies. Let h be a homeomorphism from U onto V. Choose scalars a and b from F such that $0 < |a| < 1$ and $0 < |b| < 1$. Endow $U \cup V$ with its relative topology and define the map $A\colon C(U \cup V, F) \to C(U \cup V, F)$ as follows: For $x \in C(U \cup V, F)$ and $t \in U \cup V$,

$$(Ax)(t) = \begin{cases} ax(t) + x(h(t)), & t \in U, \\ x(h^{-1}(t)) + bx(t), & t \in V. \end{cases}$$

We show that A is a surjective linear isometry.

As h and its inverse map are homeomorphisms, it is clear that Ax is a continuous map. Since a and b each have magnitudes which are strictly less than 1, A is an isometry.

A is surjective. Consider the map $D\colon C(U \cup V, F) \to C(U \cup V, F)$ defined below: For t in $U \cup V$ and x in $C(U \cup V, F)$,

$$Dx(t) = (ab - 1)^{-1}[bx(t) - x(h(t))] \quad \text{for } t \text{ in } U,$$
$$Dx(t) = (ab - 1)^{-1}[-x(t) + ax(h(t))] \quad \text{for } t \text{ in } V.$$

D is an isometry since $|ab - 1| = 1$. Since $AD = DA = 1$, A is surjective.

A is not a Stone-Banach map. The characteristic functions k_U and k_V have disjoint cozero sets but

$$Ak_U(t) = \begin{cases} ak_U(t) + k_U(h(t)) = a, & t \in U, \\ k_U(h^{-1}(t)) + bk_U(t) = 1, & t \in V, \end{cases}$$

and

$$Ak_V(t) = \begin{cases} 1, & t \in U, \\ b, & t \in V, \end{cases}$$

so $c(Ak_U) = U \cup V = c(Ak_V)$.

Extending A. A may be extended to a non-Stone-Banach map B of $C(T, F)$ onto $C(T, F)$ by defining, for $x \in C(T, F)$, $t \in T$,

$$(Bx)(t) = \begin{cases} x(t), & t \notin U \cup V, \\ (Ax)(t), & t \in U \cup V. \end{cases}$$

Moreover, if the topological space S is homeomorphic to T, under a map $g\colon S \to T$ say, then there exists a non-Stone-Banach map G between $C(T, F)$ and $C(S, F)$, namely $x \to Bx \circ g$.

Any discrete doubleton as well as the product of such a space with a nonempty 0-dimensional compact Hausdorff space is a compact 0-dimensional Hausdorff space which has disjoint, homeomorphic, nonempty, clopen subsets.

If F is a local field, let V be its valuation ring, let P be its maximal ideal and let $a_1, \ldots, a_n \in V$ be chosen so that V may be written as the disjoint union of the sets $a_i + P$. The disjoint residue classes $a_i + P$ are disjoint homeomorphic clopen subsets of V. The product of arbitrarily many such V with Tihonov topology is also a space with disjoint homeomorphic clopen subsets. Note also the following result of Banaschewski [**1**]: Any compact Hausdorff 0-dimensional second countable space which has no isolated points is homeomorphic to the space of 2-adic integers. Any such space therefore has disjoint, homeomorphic, clopen subsets.

How common are spaces without disjoint homeomorphic clopen sets? A topological space T is called *rigid* if the only homeomorphism mapping T onto itself is the identity map. If T is a compact Hausdorff space, it is readily shown that if T is not rigid, there exist disjoint nonempty open subsets of T which are homeomorphic. If, in addition, T is 0-dimensional, then T is not rigid if and only if there exist disjoint nonempty homeomorphic clopen subsets of T. In [**2** and **3**] it is shown that there are rigid spaces of arbitrarily large cardinality.

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