PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 2, June 1987

# A NOTE ON COCYCLES IN VON NEUMANN ALGEBRAS

KICHI-SUKE SAITO

ABSTRACT. In this note, we study the comparison theory for cocycles in von Neumann algebras. In particular, we investigate quasi-equivalent cocylces.

In [**1**], Connes and Takesaki studied a comparison theory for cocycles with respect to a given continuous group action on a von Neumann algebra. This theory will give rise, via the Connes cocycle theorem [**1**, 3.1, 3.5], to a corresponding comparison theory for weights on von Neumann algebras. Further, Muhly and the author in [**2**] proved that when a von Neumann algebra $M$ is in stardard form, there is essentially a one-to-one correspondence between invariant subspaces of an analytic subalgebra of $M$ which is determined by an action $\{\alpha_t\}_{t\in\mathbf{R}}$ and cocylces for $\{\alpha_t\}_{t\in\mathbf{R}}$.

In this note, we shall develop a comparison theory for cocycles in a von Neumann algebra, in particular, a finite von Neumann algebra, and apply it to a comparison theory for invariant subspaces in von Neumann algebras.

Let $M$ be a von Neumann algebra and let $G$ be a locally compact group. Let $\alpha\colon G \to \mathrm{Aut}(M)$ be a continuous action of $G$ on $M$. As in [**4**, 20.1], recall that a cocycle is an $s^*$-continuous function $a\colon G \ni s \to a(s) \in M$ with the properties:

$$a(st) = a(s)\alpha_s(a(t)) \quad \text{and} \quad a(s^{-1}) = \alpha_s^{-1}(a(s)^*), \qquad s,t \in G,$$

and that the set of all cocycles is denoted by $Z_\alpha(G,M)$. If $a \in Z_\alpha(G,M)$, then the elements $a(s) \in M$ are partial isometries:

$$a(s)a(s)^* = a(1), \quad a(s)^*a(s) = \alpha_s(a(1)), \qquad s \in G;$$

in particular, $a(1)$ is a projection, where 1 means the identity of $G$. For each $s \in G$, we set

$$_a\alpha_s(x) = a(s)\alpha_s(x)a(s)^*, \qquad x \in M_{a(1)},\ s \in G.$$

Then $_a\alpha\colon G \to \mathrm{Aut}(M_{a(1)})$ is a continuous action whose centralizer is denoted by $M^a = (M_{a(1)})^{a^\alpha}$. If $p \in \mathrm{Proj}(M^a)$, then the map $s \in G \to pa(s) \in M$ is a cocycle in $M$. We call it a subcocycle of $a$ and denote it by $a^p$.

Let $F_2$ be the type $I_2$-factor with the system of matrix units $\{e_{ij}\}_{1\le i,j\le 2}$. We shall identify $M \otimes F_2$ with $\mathrm{Mat}_2(M)$ in the usual way. Let $\iota\colon G \to \mathrm{Aut}(F_2)$ be the trivial action. Then $\alpha \otimes \iota\colon G \to \mathrm{Aut}(M \otimes F_2)$ is a continuous action. Given $a, b \in Z_\alpha(G,M)$, we define the balanced cocycle $c = c(a,b)$ associated with the cocycles $a$ and $b$ by

$$c(s) = a(s) \otimes e_{11} + b(s) \otimes e_{22} = \begin{pmatrix} a(s) & 0 \\ 0 & b(s) \end{pmatrix}, \qquad s \in G,$$

Received by the editors December 17, 1985 and, in revised form, April 14, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L10; Secondary 47A15.

and the set $I(a,b)$ by

$$I(a,b) = \{x \in a(1)Mb(1)\colon xb(s) = a(s)\alpha_s(x) \text{ for all } s \in G\}.$$

DEFINITION 1. We say that $a, b \in Z_\alpha(G,M)$ are equivalent and write $a \simeq b$ if there exists an element $c \in M$ such that

$$a(s) = c^* b(s)\alpha_s(c), \quad b(s) = ca(s)\alpha_s(c^*), \qquad s \in G.$$

We write $a \lesssim b$ if $a \simeq b^p$ for some $p \in \mathrm{Proj}(M^b)$. Further, we say that $a$ and $b$ are disjoint and write $a \overset{|}{\circ} b$ if $a(1) \otimes e_{11}$ and $b(1) \otimes e_{22}$ are centrally orthogonal in $M_2^c$. We say also that $a$ and $b$ are quasi-equivalent and write $a \sim b$ if $a(1) \otimes e_{11}$ and $b(1) \otimes e_{22}$ have the same central support in $M_2^c$. We write $a \precsim k$ if $a \sim b^p$ for some $p \in \mathrm{Proj}(M^b)$.

Let $a, b \in Z_\alpha(G,M)$. If $a \overset{|}{\circ} b$, then $I(a,b) = \{0\}$ (cf. [**4**, Proposition 20.2]). We now study the structure of $I(a,b)$ when $a$ and $b$ are not disjoint. Assume that $I(a,b) \neq \{0\}$. Let $x \in I(a,b)$, $x \neq 0$. If $x = v|x|$ is the polar decomposition of $x$, then $|x| \in I(b,b) = M^b$ and $v \in I(a,b)$. Thus we have $v^*v \leq b(1)$ and $vv^* \leq a(1)$. We put

$$p(a,b) = \sup\{v^*v\colon v \text{ is a partial isometry in } I(a,b)\}$$

and

$$q(a,b) = \sup\{vv^*\colon v \text{ is a partial isometry in } I(a,b)\},$$

respectively. Hence it is clear that the $\sigma$-weakly closed linear span $J$ of $I(a,b)^*I(a,b)$ is a $\sigma$-weakly closed 2-sided ideal of $M^b$. Thus there exists a central project $e_0$ in $M^b$ such that $J = M^b e_0$. By the definition of $p(a,b)$, we easily have $p(a,b) = e_0$ and so $p(a,b)$ is a central projection in $M^b$. Furthermore, $p(a,b)$ is the least central projection in $M^b$ such that $xp(a,b) = x$ $(x \in I(a,b))$. Similarly $q(a,b)$ is the least central projection in $M^a$ such that $q(a,b)x = x$ for all $x \in I(a,b)^* = I(b,a)$. For simplicity, we put $p(a,b) = p$ and $q(a,b) = q$, respectively.

PROPOSITION 2. *Let $a, b \in Z_\alpha(G,M)$. Then $a^{a(1)-q} \overset{|}{\circ} b$ and $a^q \sim b^p$.*

PROOF. Put $r = a(1) - q$. Then it is sufficient to prove that $I(a^r, b) = \{0\}$. Since $r \in M^a$, $a(t)\alpha_t(r)a(t)^* = r$. Let $x \in I(a^r, b)$. Then

$$\begin{aligned} xb(t) &= ra(t)\alpha_t(x) = a(t)\alpha_t(r)a(t)^*a(t)\alpha_t(x) \\ &= a(t)\alpha_t(r)\alpha_t(a(1))\alpha_t(x) = a(t)\alpha_t(rx) = a(t)\alpha_t(x). \end{aligned}$$

Thus $I(a^r, b) \subset I(a,b) \cap rMb(1) = \{0\}$. This implies that $a^r \overset{|}{\circ} b$.

Next we shall prove that $a^q \sim b^p$. To prove this, it is sufficient to prove that $q \otimes e_{11}$ and $p \otimes e_{22}$ have the same central support in $M_2^c$. Let $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ be the central support of $q \otimes e_{11} = \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $M_2^c$. By [**4**, Proposition 20.2], it is clear that $x$ (resp. $y$) is a central projection in $M^a$ (resp. $M^b$), $y = z = 0$, and $xd = dw$ for all $d \in I(a,b)$. On the other hand,

$$\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} x & 0 \\ 0 & w \end{pmatrix} \leq \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}$$

and so $q = x$ and $w \leq p$. Since $d = qd = dw$ for all $d \in I(a,b)$ and $p$ is the least central projection in $M^b$ such that $dp = d$ for all $d \in I(a,b)$, $p = w$. Thus the central support of $q \otimes e_{11}$ is $q \otimes e_{11} + p \otimes e_{22}$. Similarly, we have the central support of $p \otimes e_{22} = q \otimes e_{11} + p \otimes e_{22}$. Thus $a^q \sim b^p$. This completes the proof.

If $M$ is $\sigma$-finite and $a, b \in Z_\alpha(G, M)$ are of infinite multiplicity, then $a \simeq b$ is equivalent to $a \sim b$ [**4**, 20.2]. However, if $a, b \in Z_\alpha(G, M)$ are not necessarily of infinite multiplicity, then we have the following theorem.

THEOREM 3. *Let $a, b \in Z_\alpha(G, M)$. If $a \precsim b$, then there is a family $\{v_\gamma\}_{\gamma\in\Gamma}$ of partial isometries in $I(a, b)$ with the following properties:*
(1) $v_\gamma^* v_\gamma = 0$ *if* $\gamma \neq \lambda$;
(2) $\sum_{\gamma\in\Gamma} v_\gamma v_\gamma^* = q$;
(3) $I(a, b) = \sum_{\gamma\in\Gamma} v_\gamma M^b$,
*i.e. each $x \in I(a, b)$ can be written as $\sum_{\gamma\in\Gamma} v_\gamma x_\gamma$ for some $x_\gamma \in M^b$, where the sum converges in the $\sigma$-weak operator topology. In this case, we have $a(t) = \sum_{\gamma\in\Gamma} v_\gamma b(t)\alpha_t(v_\gamma^*)$.*

PROOF. By Zorn's lemma, there exists a maximal family $\{v_\gamma\}_{\gamma\in\Gamma}$ of partial isometries in $I(a, b)$ such that $v_\gamma^* v_\lambda = 0$ $(\gamma \neq \lambda)$. Then we can prove that $\sum_{\gamma\in\Gamma} v_\gamma v_\gamma^* = q$. Assume that $q_0 = q - \sum_{\gamma\in\Gamma} v_\gamma v_\gamma^* \neq 0$. Then, by the definition of $q$, there exists a partial isometry $v$ in $I(a, b)$ such that $vv^* q_0 \neq 0$. By the Comparability Theorem, there are a central projection $z$ in $M^a$ and partial isometries $u_1, u_2$ in $M^a$ such that $u_1^* u_1 = zq_0$, $u_1 u_1^* \leq zvv^*$, $u_2^* u_2 = (q-z)vv^*$, and $u_2 u_2^* \leq (q-z)q_0$. Then we have either $u_1 \neq 0$ or $u_2 \neq 0$. If $u_1 \neq 0$, then we set $v_1 = u_1^* zv$. Hence

$$v_1 v_1^* = u_1^* zvv^* zu_1 = u_1^* u_1 u_1^* u_1 = u_1^* u_1 = zq_0 \leq q_0.$$

Similarly, if $u_2 \neq 0$, then we put $v_2 = u_2(q - z)v$. In both cases, we have a contradiction. Thus $q = \sum_{\gamma\in\Gamma} v_\gamma v_\gamma^*$. Further, for each $x \in I(a, b)$, $x = qx = \sum_{\gamma\in\Gamma} v_\gamma v_\gamma^* x$. Since $v_\gamma^* x \in I(a, b)^* I(a, b) \subset I(b, b) = M^b$, put $x_\gamma = v_\gamma^* x$. Then $x = \sum_{\gamma\in\Gamma} v_\gamma x_\gamma$ and so $I(a, b) = \sum_{\gamma\in\Gamma} v_\gamma M^b$. Finally, we have

$$\begin{aligned}\sum_{\gamma\in\Gamma} v_\gamma b(t)\alpha_t(v_\gamma^*) &= \sum_{\gamma\in\Gamma} a(t)\alpha_t(v_\gamma)\alpha_t(v_\gamma^*) = \sum_{\gamma\in\Gamma} a(t)\alpha_t(v_\gamma v_\gamma^*) \\ &= a(t)\alpha_t\left(\sum_{\gamma\in\Gamma} v_\gamma v_\gamma^*\right) = a(t)\alpha_t(a(1)) = a(t)a(t)^* a(t) = a(t).\end{aligned}$$

This completes the proof.

Next we study the special case of Theorem 3.

THEOREM 4. *Let $M$ be a finite von Neumann algebra and $a, b \in Z_\alpha(G, M)$. Suppose that the center $\mathfrak{Z}(M^b)$ of $M^b$ is contained in the center $\mathfrak{Z}(M)$ of $M$. If $a \sim b$, then $a \simeq b$.*

PROOF. Consider a maximal family $\{u_\gamma\}_{\gamma\in\Gamma}$ of partial isometries of $I(a, b)$ such that $u_\gamma u_\gamma^*$ are mutually orthogonal and $u_\gamma^* u_\gamma$ are mutually orthogonal. Put $v = \sum_{\gamma\in\Gamma} u_\gamma$. Then $v$ is a partial isometry of $I(a, b)$. Since $q = a(1)$, suppose that $q - vv^* \neq 0$. From the definition of $q$, as in the proof of Theorem 3, there exists a partial isometry $v_1$ in $I(a, b)$ such that $v_1 v_1^* \leq q - vv^*$. Since $M$ is a finite von Neumann algebra, it is clear that $p = q = b(1)$. Let $T$ (resp. $T_0$) be the center valued trace of $M$ (resp. $M^b$). Since $\mathfrak{Z}(M^b) \subset \mathfrak{Z}(M)$, the restriction of $T$ to $M^b$ equals $T_0$. Hence we have

$$\begin{aligned}T_0(q - v^* v) &= T(q - v^* v) = T(q - vv^*) \geq T(v_1 v_1^*) \\ &= T(v_1 v_1^*) = T_0(v_1^* v_1).\end{aligned}$$

By [**5**, p. 314, Corollary 2.8], $v_1^*v_1 \precsim q - v^*v$ in $M^b$. Thus there is a partial isometry $u$ in $M^b$ such that $u^*u = v_1^*v_1$ and $uu^* \leq q - v^*v$. Put $v_2 = v_1u^*$. Then $v_2^*v_2 = uv_1^*v_1u^* = uu^*uu^* \leq q - v^*v$ and $v_2^*v_2 = v_1u^*uv_1^* = v_1v_1^* \leq q - vv^*$. Since $v_2$ is a nonzero partial isometry in $I(a,b)$, we have a contradiction. Thus $vv^* = v^*v = q$. Then $I(a,b) = vM^b$ and so $a \simeq b$. This completes the proof.

COROLLARY 5. *Let $M$ be a finite von Neumann algebra and $a \in Z_\alpha(G,M)$. If $M^\alpha$ is a factor, then $a \underset{\circ}{|} 1$ or $a \precsim 1$. Further, if $a$ is a unitary cocycle of $M$, then $a \underset{\circ}{|} 1$ or $a \simeq 1$.*

Finally we consider the form of invariant subspaces. We refer the reader to [**2**] for the definitions and notations about invariant subspaces. Let $M$ be a von Neumann algebra acting on the noncommutative Lebesgue space $L^2(M)$ in the sense of Haagerup (see [**6**]). Let $\{\alpha_t\}_{t\in\mathbf{R}}$ be a $\sigma$-weakly continuous, one-parameter group of $^*$-automorphisms of $M$. Then there is a uniquely unitary group $\{U_t\}_{t\in\mathbf{R}}$ on $L^2(M)$ such that $R_{\alpha_t(x)} = U_tR_xU_t^*$ for all $x \in M$ and $t \in \mathbf{R}$. In this note, we consider the version of right-invariant subspaces. By [**2**, Theorem 3.1], we have the following theorem.

THEOREM 5. *Let $\mathfrak{M}$ be a right-pure, right-invariant subspace of $L^2(M)$ that is left-normalized (resp. right-normalized). Then there are a projection $p$ in $M$, a strongly continuous unitary group $\{V_t\}_{t\in\mathbf{R}}$ of $\mathbf{R}$ on $L_pL^2(M)$, and $a \in Z_\alpha(G,M)$ such that*

*(1) $R_{\alpha_t(x)}L_p = V_tR_xV_t^*$ for all $x \in M,\ t \in \mathbf{R}$;*

*(2) $V_t = L_{a(t)}U_t$ for all $t \in \mathbf{R}$;*

*(3) $\mathfrak{M} = F[0,\infty)L_pL^2(M)$ (resp. $\mathfrak{M} = F(0,\infty)L_pL^2(M)$), where $F$ is the spectral measure for $V$ on $L_pL^2(M)$.*

Let $\mathfrak{M}$ be a left-normalized, right-pure, right-invariant subspace of $L^2(M)$ with $a \in Z_\alpha(\mathbf{R},M)$. If $a \precsim 1$, by Theorem 3, then there exists a family $\{v_\gamma\}_{\gamma\in\Gamma}$ of partial isometries in $I(a,1)$ such that $a(t) = \sum_{\gamma\in\Gamma} v_\gamma\alpha_t(v_\gamma^*)$. Then we have

$$\sum_{\gamma\in\Gamma} L_{v_\gamma}U_tL_{v_\gamma^*} = \sum_{\gamma\in\Gamma} L_{v_\gamma}L_{\alpha_t(v_\gamma^*)}U_t = L_{a(t)}U_t = V_t.$$

By the uniqueness of spectral decomposition, we have

$$F[\lambda,\infty) = \sum_{\gamma\in\Gamma} L_{v_\gamma}P[\lambda,\infty)L_{v_\gamma^*},$$

in particular, $F[0,\infty) = \sum_{\gamma\in\Gamma} L_{v_\gamma}P[0,\infty)L_{v_\gamma^*}$, where $P$ is the spectral measure of $\{U_t\}_{t\in\mathbf{R}}$. Hence

$$\begin{aligned}\mathfrak{M} = F[0,\infty)L^2(M) &= \sum_{\gamma\in\Gamma} L_{v_\gamma}P[0,\infty)L_{v_\gamma^*}L^2(M)\\ &= \sum_{\gamma\in\Gamma} L_{v_\gamma}P[0,\infty)L_{v_\gamma^*}L_{v_\gamma}L^2(M) = \sum_{\gamma\in\Gamma} L_{v_\gamma}L_{v_\gamma^*}L_{v_\gamma}P[0,\infty)L^2(M)\\ &= \sum_{\gamma\in\Gamma} L_{v_\gamma}P[0,\infty)L^2(M) = \sum_{\gamma\in\Gamma} \oplus L_{v_\gamma}\mathbf{H}^2,\end{aligned}$$

because $v_\gamma^*v_\gamma \in M^\alpha$ and $P[0,\infty) \in L(M^\alpha)'$. Thus we have the following proposition.

PROPOSITION 6. *Let $\mathfrak{M}$ be a left-normalized (resp. right-normalized), right-pure, right-invariant subspace of $L^2(M)$ with $a \in Z_\alpha(\mathbf{R}, M)$. If $a \precsim 1$, then there exists a family $\{v_\gamma\}_{\gamma\in\gamma}$ of partial isometries of $I(a,1)$ such that $\mathfrak{M} = \sum_{\gamma\in\Gamma} \oplus L_{v_\gamma}\mathbf{H}^2$ (resp. $\sum_{\gamma\in\Gamma} \oplus L_{v_\gamma}\mathbf{H}^2_0$).*

## REFERENCES


1. A. Connes and M. Takesaki, *The flow of weights on factors of type* III, Tôhoku Math. **J. 29** (1977), 473–575.
2. P. S. Muhly and K.-S. Saito, *Analytic subalgebras of von Neumann algebras*, Canad. J. Math. (to appear).
3. K.-S. Saito, *Nonselfadjonit subalgebras associated with compact abelian group actions on finite von Neumann algebras*, Tôhoku Math. J. **34** (1982), 485–494.
4. S. Stratila, *Modular theory in operator algebras*, Abacus Press, Tunbridge, England, 1981.
5. M. Takesaki, *Theory of operator algebras*. I, Springer-Verlag, Berlin and New York, 1979.
6. M. Terp, *$L^p$-spaces associated with von Neumann algebras*, Rapport No. 3, The University of Odense, 1981.


DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NIIGATA UNIVERSITY, NIIGATA 950-21, JAPAN