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DAUGAVET'S EQUATION AND OPERATORS ON $L^1(\mu)$

JAMES R. HOLUB

ABSTRACT. Generalizing a result of Babenko and Pichugov, it is shown that if T is a weakly compact operator on $L^1(\mu)$, where μ is a σ-finite nonatomic measure, then $\|I+T\| = 1 + \|T\|$. A characterization of all operators T on $L^1(\mu)$ having this property is also given.

In [**2**] Daugavet proved that if T is a compact operator on $C(\mu)$, then $\|I+T\| = 1 + \|T\|$, while Babenko and Pichugov [**1**] subsequently showed the same is true for a compact operator on $L^1(\mu)$ (see [**6**] for recent extensions). In general, if X is a Banach space and $T \in \mathcal{L}(X)$, then T is said to satisfy *Daugavet's equation* [**4**] if $\|I - T\| = 1 + \|T\|$. This property of an operator arises naturally in the consideration of problems of best approximation in function spaces where it has been utilized by a number of authors (e.g. [**4**], and the references cited in [**1**]).

In a recent paper [**5**] it was shown that Daugavet's equation actually holds for any weakly compact operator on $C(\mu)$. The purpose of this note is to give an analogous extension of the theorem of Babenko and Pichugov to weakly compact operators by proving that if μ is a σ-finite nonatomic measure on a space S, and T is a weakly compact operator on $L^1(\mu)$, then $\|I+T\| = 1 + \|T\|$ (Theorem 1). Related results concerning extensions and generalizations of this theorem, including a characterization of all operators T on $L^1(\mu)$ for which $\|I+T\| = 1+\|T\|$ (Theorem 2) are also given.

We begin with a simple result concerning the norming of operators on $L^1(\mu)$. As usual we denote by $\mathcal{L}(X)$ the space of all bounded linear operators on a Banach space X, by χ_A the characteristic function of a measurable set $A \subset S$, and by $\|f\|_1$ and $\|g\|_\infty$, the norms of functions $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$, respectively. The complement of a set A in S is denoted by A'.

PROPOSITION 1. *Let $T \in \mathcal{L}(L^1(\mu))$. Then given any $\varepsilon > 0$ there is a measurable set $A \subset S$ for which $0 < m(A) < \varepsilon$ and $\|T(\chi_A/m(A))\|_1 > \|T\| - \varepsilon$.*

PROOF. Since μ is σ-finite, $(L^1(\mu))^* = L^\infty(\mu)$. Therefore if $T \in \mathcal{L}(L^1(\mu))$, then $T^* \in \mathcal{L}(L^\infty(\mu))$ and $\|T\| = \|T^*\| = \sup_{\|g\|_\infty=1} \|T^*g\|_\infty$. Hence given any $\varepsilon > 0$ there is a function $g \in L^\infty(\mu)$ for which $\|g\|_\infty = 1$ and $\|T^*g\|_\infty > \|T\| - \varepsilon/2$. Since $\|T^*g\|_\infty = \operatorname{ess\,sup}_t |(T^*g)(t)|$ it follows that there is a measurable set $A \subset S$ for which $m(A) > 0$ and $|(T^*g)(t)| > \|T\| - \varepsilon$ for all $t \in S$. Replacing A (if necessary) by a subset $B \subset A$ for which $0 < m(B) < \varepsilon$ and on which the sign of $(T^*g)(t)$ is constant (which may be done since μ is nonatomic), and replacing g by $(-g)$ if

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this sign is negative, *we may assume* $0 < m(A) < \varepsilon$ *and* $(T^*g)(t) > \|T\| - \varepsilon$ *for all* $t \in A$. It follows that

$$\left\|T\left(\frac{\chi_A}{m(A)}\right)\right\|_1 \geq \left\langle g, T\left(\frac{\chi_A}{m(A)}\right)\right\rangle = \left\langle T^*g, \frac{\chi_A}{m(A)}\right\rangle = \int_S (T^*g)(t)\frac{\chi_A(t)}{m(A)}\,dt$$
$$= \frac{1}{m(A)}\int_A (T^*g)(t)\,dt > \|T\| - \varepsilon,$$

and the proposition is proved.

Using this result we can now obtain the generalization of the theorem of Babenko and Pichugov mentioned earlier.

THEOREM 1. *If T is a weakly compact operator on $L^1(\mu)$, then* $\|I + T\| = 1 + \|T\|$.

PROOF. Suppose T is a weakly compact operator on $L^1(\mu)$. Since $\|I + T\| \leq 1 + \|T\|$, we need only prove that $\|I + T\| \geq 1 + \|T\|$.

Let $\varepsilon > 0$ be given. Since T is weakly compact the set $\{Tf \mid \|f\|_1 \leq 1\}$ is weakly sequentially compact in $L^1(\mu)$ by the Eberlein-Smulian theorem (e.g. [**3**, p. 430]), and hence the set $\{|Tf| \mid \|f\|_1 \leq 1\}$ is also a weakly sequentially compact subset of $L^1(\mu)$ [**3**, p. 293]. It follows that for the given $\varepsilon > 0$ there exists a $\delta > 0$ so that if A is any measurable subset of S with $m(A) < \delta$, then $\int_A |(Tf)(t)|\,dt < \varepsilon/4$ for all $f \in L^1(\mu)$ satisfying $\|f\|_1 \leq 1$ [**3**, p. 294].

By Proposition 1 there is a set $A \subset S$ for which $0 < m(A) < \delta$ and

$$\|T(\chi_A/m(A))\|_1 > \|T\| - \varepsilon/2.$$

Setting $f = \chi_A/m(A)$ we have $\|f\|_1 = 1$ and

$$\begin{aligned}\|f + Tf\|_1 &= \int_S |f(t) + (Tf)(t)|\,dt = \int_A |f(t) + (Tf)(t)|\,dt + \int_{A'} |(Tf)(t)|\,dt \\ &\geq \int_A |f(t)|\,dt - \int_A |(Tf)(t)|\,dt + \int_{A'} |(Tf)(t)|\,dt \\ &= \|f\|_1 + \int_S |(Tf)(t)|\,dt - 2\int_A |(Tf)(t)|\,dt \\ &\geq 1 + \left(\|T\| - \frac{\varepsilon}{2}\right) - \frac{2\varepsilon}{4} = 1 + \|T\| - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary it follows that $\|I + T\| \geq 1 + \|T\|$, and the theorem is proved.

We have stated and proved Theorem 1 in a straightforward manner (rather than deriving it as a corollary to a more general, if less transparent, result) in order to emphasize its simplicity and its relationship to the results of [**1**, **2**, and **5**]. We now show that one can actually characterize those operators T in $\mathcal{L}(L^1(\mu))$ for which $\|I+T\| = 1+\|T\|$ in terms of a condition on T which has an interesting relationship to the norming property of T expressed in Proposition 1. Before stating and proving the characterization we note the following simple (and probably well known) result concerning the norm of an adjoint operator.

PROPOSITION 2. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then* $\|T^*\| = \sup_{f \in E} \|T^*f\|$, *where E denotes the set of extreme points of the unit ball of X^*.*

PROOF. By definition, $\|T^*\| \geq \sup_{f\in E}\|T^*f\|$. Suppose $\|T^*\| > \sup_{f\in E}\|T^*f\| + \varepsilon$ for some $\varepsilon > 0$. Choose $f_0 \in X^*$ and $x_0 \in X$ so that $\|f_0\| = \|x_0\| = 1$ and $|\langle T^*f_0, x_0\rangle| > \|T^*\| - \varepsilon/4$. By the Krein-Milman theorem [**3**, p. 440] there is a convex combination $\sum_{i=1}^n \lambda_i f_i$ of members of E for which $|\langle f_0 - \sum_{i=1}^n \lambda_i f_i, Tx_0\rangle| < \varepsilon/4$. It follows that

$$\begin{aligned}|\langle T^*f_0, x_0\rangle| &< \frac{\varepsilon}{4} + \left|\sum_{i=1}^n \lambda_i \langle T^*f_i, x_0\rangle\right| \\ &\leq \frac{\varepsilon}{4} \sup_{1\leq i\leq n} |\langle T^*f_i, x_0\rangle| \leq \frac{\varepsilon}{4} + \sup_{f\in E}\|T^*f\|,\end{aligned}$$

so

$$\|T^*\| - \frac{\varepsilon}{4} < |\langle T^*f_0, x_0\rangle| < \frac{\varepsilon}{4} + \sup_{f\in E}\|T^*f\| < \frac{\varepsilon}{4} + (\|T^*\| - \varepsilon).$$

That is, $\|T^*\| < \|T^*\| - \varepsilon/2$, a contradiction. Therefore $\|T^*\| \leq \sup_{f\in E}\|T^*f\|$, and the proposition is proved.

Now in Proposition 1 we showed that if $T \in \mathcal{L}(L^1(\mu))$, then for any $\varepsilon > 0$ there is a set $B \subset S$ for which $0 < m(B) < \varepsilon$ and

$$\int_S \left|T\left(\frac{\chi_B}{m(B)}\right)(t)\right| dt > \|T\| - \varepsilon.$$

Thus there is *some* set $A \subset S$ (e.g. the set $A = \{t | T(\chi_B/m(B))(t) > 0\}$) for which

$$\int_A T\left(\frac{\chi_B}{m(B)}\right)(t)\,dt - \int_{A'} T\left(\frac{\chi_B}{m(B)}\right)(t)\,dt > \|T\| - \varepsilon,$$

although in general the sets A and B are unrelated. We now show that the operators $T \in \mathcal{L}(L^1(\mu))$ for which $\|I + T\| = 1 + \|T\|$ are precisely those for which the sets A and B in the above inequality can always be assumed to satisfy the relation $B \subset A$.

THEOREM 2. *An operator $T \in \mathcal{L}(L^1(\mu))$ has the property that $\|I + T\| = 1 + \|T\| \Leftrightarrow$ for every $\varepsilon > 0$ there are measurable sets A and B in S for which $B \subset A$ and*

$$\int_A T\left(\frac{\chi_B}{m(B)}\right)(t)\,dt - \int_{A'} T\left(\frac{\chi_B}{m(B)}\right)(t)\,dt > \|T\| - \varepsilon.$$

PROOF. Suppose $T \in \mathcal{L}(L^1(\mu))$ and $\|I + T\| = 1 + \|T\|$. Then $T^* \in \mathcal{L}(L^\infty(\mu))$ and $1 + \|T^*\| = 1 + \|T\| = \|I + T\| = \|I + T^*\|$, so by Proposition 2 we have $1 + \|T^*\| = \sup_{g\in E}\|g + T^*g\|_\infty$, where E is the set of extreme points of the unit ball in $L^\infty(\mu)$. But $g \in E \Leftrightarrow |g(t)| = 1$ a.e., so for any $\varepsilon > 0$ there is a corresponding function $g_0 \in L^\infty(\mu)$ for which $|g_0(t)| = 1$ a.e. and $\|g_0 + T^*g_0\|_\infty > 1 + \|T^*\| - \varepsilon/2$. Since

$$\|g_0 + T^*g_0\|_\infty = \text{ess sup}_t\, |g_0(t) + (T^*g_0)(t)|$$

there must be a set $B \subset S$ of positive measure for which

$$\inf_{t\in B} |g_0(t) + (T^*g_0)(t)| > 1 + \|T^*\| - \varepsilon.$$

It follows, since $|g_0(t)| = 1$ a.e. and $\|T^*g_0\|_\infty \leq \|T^*\|$, that $(T^*g_0)(t)$ must have the same sign as $g_0(t)$ and $|(T^*g_0)(t)| > \|T^*\| - \varepsilon$ for all $t \in B$. Hence *we may assume $g_0(t) = 1$ and $(T^*g_0)(t) > \|T^*\| - \varepsilon$ for all $t \in B$*, since there is a subset of

B with positive measure on which either g_0 or $(-g_0)$ has this property. If we set $f_0 = \chi_B/m(B)$, then

$$\int_S g_0(t)(Tf_0)(t)\,dt = \langle g_0, Tf_0\rangle = \langle T^*g_0, f_0\rangle = \int_S (T^*g_0)(t)f_0(t)\,dt$$
$$= \frac{1}{m(B)}\int_B (T^*g_0)(t)\,dt > \|T^*\| - \varepsilon$$

(since $(T^*g_0)(t) > \|T^*\| - \varepsilon$ for $t \in B$). Setting $A = \{t|g_0(t) = 1\}$ we see that $B \subset A$ and also that

$$\int_A T\left(\frac{\chi_B}{m(B)}\right)(t)\,dt - \int_{A'} T\left(\frac{\chi_B}{m(B)}\right)(t)\,dt$$
$$= \int_S T\left(\frac{\chi_B}{m(B)}\right)(t)g_0(t)\,dt = \int_0^1 (Tf_0)(t)g_0(t)\,dt > \|T^*\| - \varepsilon.$$

Since $\|T\| = \|T^*\|$ it follows that T satisfies the stated condition.

Conversely, suppose T satisfies the condition and $\varepsilon > 0$ is given. Then there are measurable sets A and B for which $B \subset A$ and

$$\int_A T\left(\frac{\chi_B}{m(B)}\right)(t)\,dt - \int_{A'} T\left(\frac{\chi_B}{m(B)}\right)(t)\,dt > \|T\| - \varepsilon.$$

Again, for simplicity of notation, let $f_0 = \chi_B/m(B)$. If we define

$$g_0(t) = \begin{cases} 1 & \text{if } t \in A, \\ -1 & \text{if } t \notin A, \end{cases}$$

then it follows that

$$\langle Tf_0, g_0\rangle = \int_S (Tf_0)(t)g_0(t)\,dt > \|T\| - \varepsilon,$$

where $g_0(t) = 1$ for all $t \in B$ since $B \subset A$. Therefore

$$\|f_0 + Tf_0\|_1 \geq \langle f_0 + Tf_0, g_0\rangle = \langle f_0, g_0\rangle + \langle Tf_0, g_0\rangle$$
$$\geq \int_S f_0(t)g_0(t)\,dt + (\|T\| - \varepsilon) = 1 + \|T\| - \varepsilon,$$

since $f_0 = \chi_B/m(B)$ and $g_0(t) = 1$ for all $t \in B$. It follows that $\|I + T\| \geq 1 + \|T\|$, and since $\|I + T\| \leq 1 + \|T\|$ we see that $\|I + T\| = 1 + \|T\|$. The theorem is completely proved.

Although Theorem 2 is generally rather awkward to apply, it is convenient to use in certain particular cases. As an example we consider the $L^1(\mu)$-analogue of a result of Franchetti and Cheney [**4**] concerning Daugavet's equation in the space $C(S)$.

Let k be a measurable subset of S for which $0 < m(k) < m(S)$ and let $R\colon L^1(\mu) \to L^1[k]$ be the restriction map. Suppose $E\colon L^1[k] \to L^1(\mu)$ is some bounded linear extension mapping. Then ER is a projection on $L^1(\mu)$ with $\|ER\| = \|E\|$. In the setting of the result of Franchetti and Cheney mentioned above k is taken instead to be a closed, nonopen, subset of S, R and E are taken to be the analogous restriction and extension operators on $C(S)$ and $C(k)$, respectively, and it is shown that for *any* such E the projection ER on $C(S)$ satisfies Daugavet's equation—i.e.

$\|I - ER\| = 1 + \|E\|$. In contrast, for the $L^1(\mu)$ case we have

PROPOSITION 3. *There is no bounded linear extension mapping $E\colon L^1[k] \to L^1(\mu)$ for which the projection ER on $L^1(\mu)$ satisfies Daugavet's equation.*

PROOF. Let $K \subset S$ for which $0 < m(k) < m(S)$, let $R\colon L^1(\mu) \to L^1[k]$ be the restriction map, and let $E\colon L^1[k] \to L^1(\mu)$ be any bounded linear extension mapping. If $\|I - ER\| = 1 + \|E\|$, then by Theorem 2 it must be true that given any $\varepsilon > 0$ there exist measurable subsets A and B of S with $B \subset A$ for which

$$\int_A (-ER)\left(\frac{\chi_B}{m(B)}\right) dt - \int_{A'} (-ER)\left(\frac{\chi_B}{m(B)}\right) dt > \|E\| - \varepsilon.$$

That is, since $R(\chi_B) = \chi_{B\cap k}$,

$$\frac{\|E\| m(B \cap k)}{m(B)} \geq \int_{A'} E\left(\frac{\chi_{B\cap k}}{m(B)}\right) dt - \int_A E\left(\frac{\chi_{B\cap k}}{m(B)}\right) dt > \|E\| - \varepsilon.$$

In particular, $m(B \cap k)/m(B) \geq 1 - \varepsilon$, since $\|E\| \geq 1$. Moreover, since $B \subset A$ we then have

$$\int_{A'} E\left(\frac{\chi_{B\cap k}}{m(B)}\right) dt - \int_{A-B} E\left(\frac{\chi_{B\cap k}}{m(B)}\right) dt - \int_B E\left(\frac{\chi_{B\cap k}}{m(B)}\right) dt > \|E\| - \varepsilon,$$

or

$$\left[\int_{A'} E\left(\frac{\chi_{B\cap k}}{m(B)}\right) dt - \int_{A-B} E\left(\frac{\chi_{B\cap k}}{m(B)}\right) dt - \int_{B-k} E\left(\frac{\chi_{B\cap k}}{m(B)}\right) dt\right]$$
$$- \frac{m(B \cap k)}{m(B)} \int_{B\cap k} E\left(\frac{\chi_{B\cap k}}{m(B \cap k)}\right) dt > \|E\| - \varepsilon.$$

Now $A', A - B$, and $B - k$ are disjoint, $\|\chi_{B\cap k}/m(B)\|_1 \leq 1$, and $m(B \cap k)/m(B) \geq 1 - \varepsilon$, so since $E(\chi_{B\cap k})(t) = \chi_{B\cap k}(t)$ for $t \in B \cap k$ it follows from this that $\|E\| - (1 - \varepsilon) > \|E\| - \varepsilon$, or $1 < 2\varepsilon$. Since $\varepsilon > 0$ arbitrarily we have reached a contradiction, and the proposition is established.

REMARKS. (1) It was observed in [**5**] that if S is a compact Hausdorff space and $T \in \mathcal{L}(C(S))$, then either $\|I + T\|$ or $\|I - T\| = 1 + \|T\|$. In particular, if μ is a positive measure and $T \in \mathcal{L}(L^\infty(\mu))$, then (since $L^\infty(\mu)$ is isometric to some such $C(S)$ [**3**, p. 445]) either $\|I+T\|$ or $\|I-T\| = 1+\|T\|$. Consequently, if $T \in \mathcal{L}(L^1(\mu))$, then $T^* \in \mathcal{L}(L^\infty(\mu))$, so either $\|I + T^*\|$ or $\|I - T^*\| = 1 + \|T^*\| = 1 + \|T\|$. It follows, then, that either $\|I + T\|$ or $\|I - T\| = 1 + \|T\|$ for any $T \in \mathcal{L}(L^1(\mu))$.

(2) The condition that μ be nonatomic is necessary for the validity of our results. For example, if $T = (-1)e_1 \otimes e_1$ in $\mathcal{L}(l^1)$, then T is compact; yet

$$\|I + T\| = \sup_n \|e_n - \langle e_1, e_n\rangle e_1\| = 1 \neq 1 + \|T\|.$$

In the same way, if the measure space (S, Σ, μ) has an atom $A \in \Sigma$, then by a similar construction there is a one-dimensional operator $T \in \mathcal{L}(\mu)$ for which $\|I + T\| \neq 1 + \|T\|$.

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DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061