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PROOF OF A CONJECTURE OF BANK AND LAINE REGARDING THE PRODUCT OF TWO LINEARLY INDEPENDENT SOLUTIONS OF $y'' + Ay = 0$

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Dedicated to Professor A. Edrei on his 70th birthday

ABSTRACT. Let A be a transcendental entire function of order < 1. If w_1 and w_2 are two linearly independent solutions of the differential equation $y'' + Ay = 0$, then at least one of w_1, w_2 has the property that the exponent of convergence of its zeros is > 1.

1. Introduction. In this note we consider the differential equation

$$y'' + A(z)y = 0, \tag{1}$$

where A is entire. Let w_1 and w_2 be two linearly independent solutions of the D.E. (1). Assume that w_1 and w_2 are normalized so that their Wronskian is identically 1. That this is possible is an obvious consequence of the homogeneous character of (1). Set $f = w_1 w_2$. Bank and Laine [2] observed that the function f satisfies the differential equation

$$-4Af^2 = 2ff'' - (f')^2 + 1. \tag{2}$$

From the above relation, it follows that f has the property: *If z_0 is a zero of f, then either $f'(z_0) = 1$ or $f'(z_0) = -1$. Then we shall say that f has the B-L property at z_0.*

If $f(z)$ has the B-L property at each one of its zeros, we simply say that $f(z)$ has the B-L property.

Using the Wiman-Valiron theory, Bank and Laine [2, p. 358] proved that if the order of f is $< \frac{1}{2}$, then f cannot have the B-L property.

Edrei [4] proved that if the growth of the Nevanlinna characteristic of an entire, transcendental function f is sufficiently regular, then it is impossible for f to be of order < 1 and also possess the B-L property.

The main purpose of this note is to establish

THEOREM 1 (BANK-LAINE CONJECTURE). *An entire function f, of order < 1, cannot possess the B-L property unless it is a polynomial of the form $az^2 + bz + c$ with $4ac - b^2 + 1 = 0$.*

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Instead of proving Theorem 1 directly, we shall derive it from the following slightly more informative

THEOREM 2. *Let f be an entire function of order ρ, and of lower order λ, possessing the B-L property.*

(i) *If f is transcendental, then $\rho + \lambda \geqslant 2$.*

(ii) *If f is not transcendental, then f is a polynomial of degree $\leqslant 2$, of the form $az^2 + bz + c$ with $4ac - b^2 + 1 = 0$.*

It should be noted that assertion (i) of the theorem is automatically satisfied if $\rho \geqslant 2$; it is therefore sufficient to carry out our proofs under the restriction $\rho < 2$.

The following consequence of Theorem 2 is immediate.

COROLLARY 1. *Entire functions of order* 1 *possessing the B-L property must have regular growth* ($i.e.$, $\rho = \lambda$).

Theorem 1 may also be translated into an equivalent statement about differential equations.

COROLLARY 2. *Let $A(z)$ be entire, transcendental, and of order < 1, and let $w_1(z)$, $w_2(z)$ be two linearly independent solutions of $y'' + Ay = 0$. Then at least one of them has zeros whose exponent of convergence is > 1.*

Our method is based on a consequence of Carleman's differential inequality, stated below as Arima's Theorem A.

Let $D = \{z\colon |f(z)| > 1\}$. For any $r > 0$, let D_r be the part of D lying in $|z| < r$. Let $A_k(r)$ $(k = 1, 2, \ldots, n(r))$ be the arcs of $|z| = r$ contained in D, and let $r\theta_k(r)$ be their lengths. We define $\theta(r) = \infty$ if the entire circle $|z| = r$ lies in D. Otherwise, $\theta(r) = \max_k \theta_k(r)$.

THEOREM A (ARIMA). *Let f be entire, and let D be the domain where $|f(z)| > 1$. Let $\theta(r)$ be defined as above for the domain D. Then for any $0 < \alpha < 1$ we have*

$$\ln \ln M(r, f) > \pi \int_{r_0}^{\alpha r} \frac{dt}{t\theta(t)} - c(\alpha, r_0), \tag{3}$$

where $0 < r_0 < \alpha r$ and $c(\alpha, r_0)$ is independent of r.

The proof can be found in Arima's paper [**1**, p. 64].

REMARK. Without altering Arima's proof, it is easily seen that if f is an analytic function, single-valued and regular in a region $|z| > R_0 > 0$, and if, as $r \to \infty$,

$$\max_{|z|=r} |f(z)| = M(r, f) \to \infty, \qquad r > R_0,$$

then the inequality (3) of Theorem A continues to hold provided r_0 is large enough.

Our proof requires this slightly extended version of Arima's theorem.

2. Functions with the B-L property. The following two lemmas characterize functions with the B-L property.

LEMMA 2.1. *Let f be an entire function. Define*

$$h(z) = 1/f^2 - (f'/f)^2 + 2f''/f. \tag{2.1}$$

If f has the B-L property, then h is entire.

LEMMA 2.2. *Let f be an entire function with the B-L property. Then $f = w_1 w_2$, where w_1 and w_2 are linearly independent solutions of the D.E.* (1) *with A defined as in* (2).

The proofs of both lemmas can be found in Bank and Laine's paper [3, p. 666].

3. Estimates for f''/f and f'/f. The following lemma plays an essential role in the proof of Theorem 2.

LEMMA 3.1. *Let f be entire, of order ρ, $\rho < 2$, and $I_k = (2^k, 2^{k+1})$. Then for all k large, say $k > k_0$, there exists a set E_k in I_k which is a finite union of subintervals, and such that*

$$\mu(E_k) < (4/5k \ln 2)\mu(I_k), \tag{3.1}$$

and

$$\left|\frac{f''(re^{i\theta})}{f(re^{i\theta})}\right| + \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^2 < Mr^6, \qquad 0 \leqslant \theta < 2\pi, \tag{3.2}$$

if $r \in I_k - E_k$, where M is a constant independent of k.

(By $\mu(A)$ we denote the linear length of the set A.)

PROOF. Write

$$f(z) = Kz^N e^{cz} \prod \left(1 - \frac{z}{a_n}\right) e^{z/a_n}, \qquad 0 < |a_1| \leqslant |a_2| \leqslant |a_3| \leqslant \cdots.$$

Differentiating f logarithmically and noting the identity $f''/f = (f'/f)' + (f'/f)^2$, we obtain

$$\frac{f'}{f} = c + \frac{N}{z} + \sum_{j=1}^{\infty} \frac{1}{a_j} + \frac{1}{z - a_j}, \tag{3.3}$$

$$\frac{f''}{f} = \left(\frac{f'}{f}\right)^2 - \frac{N}{z^2} - \sum_{j=1}^{\infty} \frac{1}{(z - a_j)^2}. \tag{3.4}$$

Let $n(t)$ be the usual counting function of the zeros of f. Since the order of f is < 2, we have, for all t large, say $t > t_0$, $n(t) < t^{\xi}$, where $\rho < \xi < 2$.

Choose an integer k_0 so that $2^{k_0} \geqslant t_0$. For an integer $k \geqslant k_0$, let $r_k = 2^k$ and $m = n(3r_k)$. Then

$$m < (er_k)^{\xi}. \tag{3.5}$$

We now confine z to the annulus

$$r_k \leqslant |z| \leqslant r_{k+1}. \tag{3.6}$$

We note that (3.3) yields

$$\frac{f'}{f} = z \sum_{j=1}^{m} \frac{1}{a_j(z - a_j)} + R_1(z) \tag{3.7}$$

with

$$|R_1(z)| \leqslant 3|z| \sum_{j=m+1}^{\infty} \frac{1}{|a_j|^2} + |c| + \frac{N}{|z|} = o(|z|). \tag{3.8}$$

Similarly, from (3.4)

$$\frac{f''}{f} = \left(\frac{f'}{f}\right)^2 - \sum_{j=1}^{m} \frac{1}{(z-a_j)^2} + R_2(z) \tag{3.9}$$

with

$$|R_2(z)| \leqslant 9 \sum_{j=m+1}^{\infty} \frac{1}{|a_j|^2} + \frac{N}{|z|^2} = O(1). \tag{3.10}$$

To complete our estimate we use Cartan's lemma in geometrical form. Set

$$d = r_k/5 \log r_k, \qquad k \geqslant k_0. \tag{3.11}$$

Then by Cartan's lemma there exist p disks, $p \leqslant m$. (See [5, pp. 20–21].)

$$D_j = \left\{ z\colon |z - a_j| \leqslant d_j \right\}, \qquad j = 1, 2, \ldots, p, \tag{3.12}$$

such that $\sum_{j=1}^{p} d_j = 2d$ and such that, if $z \notin \bigcup_{j=1}^{p} D_j$, it is possible to find a renumbering $a_1', a_2', \ldots, a_m'$ of $a_1, a_2, \ldots, a_m$, satisfying

$$\left|z - a_j'\right| \geqslant jd/m, \qquad j = 1, 2, \ldots, m. \tag{3.13}$$

(The renumbering may depend on z.)

From (3.13), (3.5), and (3.6), we obtain

$$\left|\sum_{j=1}^{m} \frac{z}{a_j(z-a_j)}\right| \leqslant |z| \left(\frac{m}{d}\right) \sum_{j=1}^{m} \frac{1}{|a_j|} = \frac{m|z|}{d} \int_0^{3r_k} \frac{dn(t)}{t} = o\left(|z|^3\right). \tag{3.14}$$

$$\left|\sum_{j=1}^{m} \frac{1}{(z-a_j)^2}\right| = O\left(\frac{m^2}{d^2}\right) = o\left(|z|^2\right) \tag{3.15}$$

for $r_k < |z| < r_{k+1}$ and $z \notin \bigcup_{j=1}^{p} D_j$.

Let A_j be the annulus generated by revolving the disks D_j around the origin, and let E_k^* be the intersection of $\bigcup_{j=1}^{p} A_j$ with the positive real axis. Now let

$$E_k = (2^k, 2^{k+1}) \cap E_k^*.$$

Since the sum of the diameters of D_j is $4d$, the length of E_k is not greater than $4d$. This proves (3.1).

Clearly, (3.2) follows from (3.7)–(3.10), (3.14), and (3.15).

For each $k \geqslant k_0$ we can construct a set E_k in the above manner. We then define $E = \bigcup_{k=k_0}^{\infty} E_k$ and $\tilde{E} = \{re^{i\theta}\colon r \in E; 0 \leqslant \theta \leqslant 2\pi\}$.

From (3.1) we easily derive

LEMMA 3.2. *Let r_0 be any fixed positive number and, for each $r > 0$, let J_r be the set defined as $J_r = [r_0, \alpha r] - E$, $0 < \alpha < 1$. Then*

$$\lim_{r \to \infty} \frac{1}{\ln r} \int_{J_r} \frac{1}{t}\, dt = 1.$$

4. Proof of Theorem 2. Since f has the B-L property, Lemma 2.1 implies that the function h defined as (2.1) is entire. And since the order of f is ρ, the order of h must also be $\leqslant \rho$. Now there are two cases to be considered. We first consider

Case 1. h is a polynomial of degree N. From Lemma 2.2, f is a product of two linearly independent solutions of D.E.

$$y'' - \tfrac{1}{4}hy = 0. \tag{4.1}$$

Using Wiman-Valiron's theory of the maximum term, Bank and Laine [**2**, p. 354] observed that if the degree $N \geqslant 1$, then

$$\ln M(r, f) = c_1 r^{(N+1)/2}(1 + o(1)) \quad \text{as } r \to \infty, \tag{4.2}$$

for some constant $c_1 > 0$. From (4.2) we conclude that f has regular growth, that is, $\rho = \lambda = (N+2)/2$. Thus $\rho + \lambda = N + 2 \geqslant 3$. We now consider the case $N = 0$. Then h is constant. It is easy to see that if h is a nonzero constant, then either f is a nonzero constant or f satisfies (4.2) with $N = 0$, and this implies that $\rho + \lambda = 2$. If $h = 0$, then (2.1) becomes

$$2ff'' + 1 - (f')^2 = 0. \tag{4.3}$$

To solve f, we differentiate (4.3) and find $ff''' = 0$. This implies that either f is identically zero or f is a polynomial of degree $\leqslant 2$. Since f satisfies (4.3), f has to be of the form $az^2 + bz + c$ with $4ac - b^2 + 1 = 0$.

We now turn to

Case 2. h is transcendental. Let

$$D_1 = \{z \colon |f(z)| > 1\}, \qquad D_2 = \{z \colon |h(z)/z^7| > 1\},$$

and let $\theta_i(r)$ be defined as in §1 for the domain D_i, $i = 1, 2$. From (2.1) and (3.2) we deduce that for $z \in D_1 - \tilde{E}$,

$$|h(z)| < 1 + Mr^6 < r^7, \qquad |z| = r > r_0. \tag{4.4}$$

On the other hand, for $z \in D_2 - \tilde{E}$ we have

$$|h(z)| > h^7. \tag{4.5}$$

Therefore, from (4.4) and (4.5) we deduce

LEMMA 4.1. *Let* $\theta(r) = \theta_1(r) + \theta_2(r)$. *Then*

$$\theta(r) < 2\pi \tag{4.6}$$

and

$$\theta_1(r)\theta_2(r) \leqslant \theta^2(r)/4, \tag{4.7}$$

for $r \geqslant r_0$ *and* $r \notin E$.

In view of the following remark, we may assume that $\theta_1(r)\theta_2(r) \neq 0$ for all $r \geqslant r_0$ and $r \notin E$.

REMARK. If there exists a sequence $\{t_n\}$ tending to infinity such that
(i) $\theta_1(t_n) = 0$, then f is clearly a constant, and from (2.1) h is a constant;
(ii) $\theta_2(t_n) = 0$, then h is a polynomial.
We have already discussed this in Case 1.

From Theorem A and the remark following it we have

$$\ln \ln M(r,f) > \pi \int_{r_0}^{\alpha r} \frac{dt}{t\theta_1(t)} - k_1, \tag{4.8}$$

$$\ln \ln M\left(r, \frac{h}{z^7}\right) > \pi \int_{r_0}^{\alpha r} \frac{dt}{t\theta_2(t)} - k_2. \tag{4.9}$$

Therefore, it follows from (4.6)–(4.9) that

$$\begin{aligned} \ln \ln M(r,f) + \ln \ln M\left(r, \frac{h}{z^7}\right) &> \pi \int_{r_0}^{\alpha r} \frac{\theta(t)}{\theta_1(r)\theta_2(r)} \frac{dt}{t} + k \\ &> 4\pi \int_{J_r} \frac{1}{t\theta(r)}\, dt + k > 2\int_{J_r} \frac{dt}{t} + k, \qquad k = -k_1 - k_2. \end{aligned} \tag{4.10}$$

From Lemma 3.2 and (4.10), we conclude immediately that

$$\begin{aligned} \rho + \lambda &\geqslant \varliminf_{r\to\infty} \frac{\ln\ln M(r,f)}{\ln r} + \varlimsup_{r\to\infty} \frac{\ln \ln M(r,h)}{\ln r} \\ &\geqslant \varliminf_{r\to\infty} \left(\ln \ln M(r,f) + \ln \ln M(r, h/z^7)\right)/\ln r \\ &\geqslant 2. \end{aligned}$$

Combining the conclusions of both cases, we establish Theorem 2.

We now prove Corollary 2. Let ρ_A be the order of the function $A(z)$. By assumption $\rho_A < 1$. Let the Wronskian of w_1 and w_2 be normalized so that it is identically one. Then the product $f = w_1 w_2$ satisfies the differential equation (2). This implies that the order of f is $\geqslant \rho_A$. Let ρ be the order of f and we assume that $\rho \leqslant 1$. Replacing the function h in Case 2 of Theorem 2 by $-4A$ and repeating the same argument we conclude that $\rho_A + \rho \geqslant 2$. But this is incompatible with the assumptions that $\rho \leqslant 1$ and $\rho_A < 1$. Therefore the order of f must be > 1. This corollary follows immediately from the estimate [**2**, p. 354, (8)]

$$T(r,f) = O(N(r,1/f) + T(r,A) + \ln r) \quad \text{n.e. as } r \to \infty.$$

REMARK. It has to be pointed out that to derive (4.2) it is absolutely essential that h be a polynomial. If h is replaced by a rational function, then (4.2) is no longer true. (For a counterexample see [**2**, p. 355].)

5. An example. There are functions of order < 1 that have the B-L property at all but one of their zeros.

One such example is the function $f(z) = 2\sqrt{z}\, \sin \sqrt{z}$. It is entire, of order $1/2$, and with the exception of $z = 0$, where $f(0) = 0$ and $f'(0) = 2$, the B-L property is satisfied at all the other zeros of f. By substituting this function into (2.1) we see that f satisfies the D.E.

$$2y''/y - (y'/y)^2 + 1/y^2 = -(z+3)/4z^2 = h(z).$$

Note that in this case h is a rational function with a pole at $z = 0$. From the proof of Lemma 2.1 it is easy to see that the existence of this pole at $z = 0$ is caused by the failure of the B-L property at this point.

Using the method in the last section, we can derive the following

PROPOSITION. *Let f be an entire function of order < 1 and let $f(0) = 0$. Assume that f has the B-L property at all its zeros with the exception of $z = 0$, where $f'(0) = 2$. If the power series of f is $f(z) = 2z + c_2 z^2 + \cdots$, $c_2 \neq 0$, then*

$$f(z) = \frac{2}{\sqrt{-3c_2}} z^{1/2} \sin\sqrt{-3c_2 z}\,.$$

We note that if $c_2 = 0$, then $f(z) = 2z$.

OUTLINE OF THE PROOF. A straightforward substitution of f into (2.1) yields

$$2f''/f - (f'/f)^2 + 1/f^2 = -\tfrac{3}{4}(z^{-2} - c_2 z^{-1}) + h_1, \tag{5.1}$$

where h_1 is an entire function.

Since the order ρ of f is < 1, we can improve the estimate of (3.2) by choosing ξ in (3.5) to be $(1 + 2\rho)/3$. We then obtain

$$\left|\frac{f''(re^{i\theta})}{f(re^{i\theta})}\right| + \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| < Mr^{\rho-1}, \qquad r \notin E. \tag{5.2}$$

Repeating the argument used in Case 2, we conclude that h_1 cannot be transcendental. From (5.1) and (5.2) we deduce further that $h_1 = 0$. Therefore, f satisfies the D.E.

$$2y''/y - (y'/y)^2 + 1/y^2 = -\tfrac{3}{4}(z^{-2} - c_2 z^{-1}). \tag{5.3}$$

One way to obtain a general solution of the above equation is to consider the D.E.

$$w'' + \tfrac{3}{16}(z^{-2} - c_2 z^{-1})w = 0. \tag{5.4}$$

Since the coefficient of w in (5.4) is a rational function with a pole at $z = 0$, the solutions of this D.E. are no longer entire. In fact, by direct substitution it is easily verified that

$$w_1^* = z^{1/4}\cos\left(\sqrt{-3c_2 z}/2\right), \tag{5.5}$$

$$w_2^* = z^{1/4}\sin\left(\sqrt{-3c_2 z}/2\right), \tag{5.6}$$

are two linearly independent solutions of the D.E. (5.4).

If w_1 and w_2 are any two linearly independent solutions of this D.E., and if they are normalized so that their Wronskian is 1, then $f = w_1 w_2$ satisfies (5.3). Conversely, if f satisfied (5.3), then f is a product of two normalized linearly independent solutions of (5.4).

Hence $f = (4/\sqrt{-3c_2})w_1^* w_2^*$ is a solution of the D.E. (5.3). From (5.5) and (5.6) we see that this function is the only solution which is entire. (All of the other solutions have a branch point at $z = 0$.) It is easy to verify that f has all the required properties, and the proof is complete.

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