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C^∞-INVARIANTS ON LOOP SPACES

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ABSTRACT. We compute all the C^∞-invariants on the spaces of immersions $C_I^\infty(S^1, S)$ and $C_I^\infty(S^1, \mathbf{R}^3)$, where S is a surface of constant curvature, by using Griffiths' formalism on the calculus of variations via exterior differential systems.

Introduction. Let M be a manifold and $C_I^\infty(S^1, M)$ the space of immersions from S^1 into M. In this paper, using the techniques of moving frames [**1**, **2**, **4**] and the calculus of variations via exterior differential systems [**4**], we compute all the C^∞-invariants (i.e. functionals which are invariant under C^∞-deformations) on $C_I^\infty(S^1, M)$ for certain manifolds M. For the case $M = \mathbf{R}^2$, the famous classical example is the winding number

$$w(\gamma) = \frac{1}{2\pi} \oint_\gamma \kappa\, ds,$$

and nothing else is known. Now by Whitney's theorem any two closed curves in $\mathbf{R}^2$ of the same winding number are homotopic to one another through immersions, so it is almost transparent that there should not be any other nontrivial C^∞-invariants, and it will be shown to be true. For the case $M = \mathbf{R}^3$, intuitively by the fact that all regular closed curves in $\mathbf{R}^3$ are homotopic to each other through immersions, there should not be any nontrivial C^∞-invariants on $C_I^\infty(S^1, \mathbf{R}^3)$ and again it will be shown that this is exactly the case. The analogous problem for the case $M = S =$ a general surface of constant curvature k is most interesting. On a general surface we do not have the notion of winding numbers as in $\mathbf{R}^2$, and of course it is not true that all closed curves are homotopic to each other through immersions as in $\mathbf{R}^3$. It turns out that if the constant curvature k is not zero, there is no nontrivial C^∞-invariant on $C_I^\infty(S^1, S)$, and if $k = 0$, then $\int_\gamma \kappa\, ds$, where κ is the geodesic curvature, is the only invariant. Two variations, namely the restrictions to $\kappa \equiv 1$ or $\tau \equiv 1$ (notice that there are many such curves) of the case $M = \mathbf{R}^3$ are also discussed. It turns out that in either case there is no nontrivial C^∞-invariant on the corresponding loop spaces.

Now if a functional $\Phi(\gamma)$ on $C_I^\infty(S^1, M)$ is a C^∞-invariant, then the Euler-Lagrange equations associated to Φ are trivial. So the problem of determining C^∞-invariants reduces to a problem in the calculus of variations determining those functionals with trivial Euler-Lagrange equations. For this we shall adopt the method of Griffiths [**4**]. Although we shall not make it explicit, these should reduce to computing certain cohomology groups of the spectral sequence discussed in [**3**].

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The author would like to thank the referee for pointing out that Proposition (II.1) below, which is proved directly using intrinsic terms, can also be proved using the result of Anderson and Duchamp (*On the existence of global variational principles*, Amer. J. Math. **102** (1980), 781–868), which says that in the case of one independent variable every Lagrangian with trivial Euler-Lagrange equations must be linear in the highest derivative involved, although this would require treating the Lagrangian ϕ in terms of local coordinates.

Throughout this paper, the summation convention will be adopted. If η is a differential form on a manifold X and $Y \subset X$ a submanifold, we shall write η_Y for $\eta|Y$ and $\eta \equiv 0 \bmod Y$ for $\eta_Y = 0$. If I is a differential ideal in $\Omega^*(X)$, then $\eta \equiv 0 \bmod I$ shall mean $\eta \in I$.

I. Preliminaries. Let X be a manifold. On X let I be a Pfaffian differential system generated by the 1-forms $\{\theta^\alpha\colon\ 1 \le \alpha \le m\}$, let ω be a 1-form called the independence condition, and let ϕ be a Lagrangian 1-form. An integral manifold N of (I, ω) is a connected submanifold of X satisfying $\theta_N = 0\ \forall \theta \in I$ and $\omega_N \neq 0$. The totality of integral manifolds of (I, ω) will be denoted by $\mathcal{I}(I, \omega)$. For each $N \in \mathcal{I}(I, \omega)$, we set $\Phi(N) = \int_N \phi$. Here we agree to consider only those N's for which the integral exists (i.e. converges) and we thus may view Φ as a functional (perhaps not everywhere defined) on $\mathcal{I}(I, \omega)$. We denote by $(X; I, \omega; \phi)$ the variational problem associated to the functional Φ.

Let $N \in \mathcal{I}(I, \omega)$ and let $\{N_t\} \subset \mathcal{I}(I, \omega)$ be a variation of N. In [**4**] it was found that the condition for the expression

$$\left[\frac{d}{dt}\int_{N_t} \phi\right]\bigg|_{t=0}$$

to be zero is

$$v \lrcorner\, d(\phi + \lambda_\alpha \theta^\alpha) \equiv 0 \mod N \tag{1}$$

for all $v \in C^\infty(N, T(X))$, where the λ_α's are functions to be determined. Equations (1) are then defined as the Euler-Lagrange equations for the variational problem $(X; I, \omega; \phi)$, and in the case that $(X; I, \omega; \phi)$ is a classical variational problem, equations (1) are just the classical Euler-Lagrange equations found in any text.

REMARK. Of course in order to make sense we should specify the boundary conditions before we compute the Euler-Lagrange equations. But in our case the integral manifolds we shall be concerned with are closed curves in a manifold M and thus no boundary conditions are needed. For the correct boundary conditions for the general case, one can refer to [**4**].

In general, to compute the Euler-Lagrange equations (1) it is sufficient to use a set of vectors that span $T_x(X)$ at every $x \in N$. More importantly, we may use "1 less" vector, as shown in the following algebraic lemma whose proof is completely trivial.

LEMMA (I.1). *Let T be a vector space and $\Psi \in \Lambda^2 T^*$. Let $v_1, \ldots, v_s$ be vectors that span an s-dimensional subspace V of T and w a vector not in V such that $\{v_i, w\}$ form a basis of T. Then if $\langle v_i \lrcorner\, \Psi, w\rangle = 0\ \forall i = 1, \ldots, s$, we have $\langle v \lrcorner\, \Psi, w\rangle = 0\ \forall v \in T$.*

REMARK. Given a manifold X, a Pfaffian system with independence condition (I, ω), and a Lagrangian 1-form ϕ, we may want to restrict the functional

$\Phi(N) = \int_N \phi$ to the set of those integral manifolds of (I, ω) satisfying certain subsidiary conditions. For this we need to restrict (I, ω) and ϕ to the submanifold $Y \subset X$ defined by these subsidiary conditions (or constraints) and consider the variational problem $(Y; I|Y, \omega|Y; \phi|Y)$ (cf. [**3**, **4**]).

II. Main results. We first consider the case $M = S$, a surface of constant curvature $k \in \mathbf{R}$. Now it is well known that on such a surface there exists a transitive Lie group G of isometries such that every curve $\gamma \subset S$ is uniquely determined by the geodesic curvature κ along γ up to a left translation by an element in G. Let γ be a regular closed curve on S, $\kappa = \kappa(s)$ the geodesic curvature of γ expressed as a function of the arc length s. Thus functionals on $C_I^\infty(S^1, S)$ are of the form $\Phi(\gamma) = \oint_\gamma f\, ds$, where $f = f(\kappa, \dot\kappa, \ldots)$ (here the over-dots denote derivatives with respect to s). Hence the problem of determining C^∞-invariant functionals on $C_I^\infty(S^1, S)$ reduces to that of determining those functions of the form $f = f(\kappa, \dot\kappa, \ldots)$ with the property that the Euler-Lagrange equations associated to the functional $\Phi(\gamma)$ ($\gamma \in C_I^\infty(S^1, S)$) are trivial. For this we have

PROPOSITION (II.1). *Let S be a surface of constant curvature k.*

(a) *If $k \neq 0$, there is no nontrivial C^∞-invariant on $C_I^\infty(S^1, S)$.*

(b) *If $k = 0$, the only nontrivial C^∞-invariants on $C_I^\infty(S^1, S)$ are constant multiples of the winding number w, i.e. $f = a\kappa + dg/ds$ for some constant $a \in \mathbf{R}$ and some function $g = g(\kappa, \dot\kappa, \ldots)$.*

PROOF. By the discussions immediately before the proposition, it suffices to show that for all $n \geq 0$, the only Lagrangian functions $f = f(\kappa, \dot\kappa, \ldots, \kappa^{(n)})$ having trivial Euler-Lagrange equations are exact (i.e. $f = dg/ds$ for some function g) if $k \neq 0$ and $f = a\kappa + dg/ds$ if $k = 0$. We shall use induction on n.

When $n = 0$, i.e. $f = f(\kappa)$, let $\mathcal{F}(S)$ be the orthonormal frame bundle on S. It is well known that $\dim \mathcal{F}(S) = 3$ and there is a coframing $\{\omega^1, \omega^2, \omega_1^2\}$ of $\mathcal{F}(S)$ satisfying the structure equations [**4**, **5**]

$$(2) \qquad \begin{cases} d\omega^1 = -\omega^2 \wedge \omega_1^2, \\ d\omega^2 = \omega^1 \wedge \omega_1^2, \\ d\omega_1^2 = -k\omega^1 \wedge \omega^2. \end{cases}$$

(For the case that S is a surface in $\mathbf{R}^3$, $\{\omega^1, \omega^2\}$ are dual to the tangential vectors in the Darboux frame of S.) We set

$$X = \mathcal{F}(S) \times \mathbf{R}, \quad \text{where } \mathbf{R} \text{ has coordinate } \kappa,$$

$$I = \begin{cases} \theta^1 = \omega^2 = 0, \\ \theta^2 = \omega_1^2 - \kappa\omega = 0, \end{cases} \quad \omega = \omega^1 = ds \neq 0, \quad \phi = f(\kappa)\omega.$$

It is clear that there is a natural (once the orientation is fixed) one-to-one correspondence between $\gamma \subset S$ and $N \in \mathcal{I}(I, \omega)$ in X, and so the problem reduces to determining those functions $f = f(\kappa)$ with the property that the Euler-Lagrange equations associated to the variational problem $(X; I, \omega; \phi)$ are trivial.

Following the prescription in §I and using the structure equations (2) we compute $\Psi = d(\phi + \lambda_\alpha \theta^\alpha)$ ($\alpha = 1, 2$). Denoting by $\{\partial/\partial\omega, \partial/\partial\theta^\alpha, \partial/\partial\kappa\}$ the dual frame of the coframe $\{\omega, \theta^\alpha, d\kappa\}$ of X and contracting the vector fields $\partial/\partial\kappa$ and $\partial/\partial\theta^\alpha$ with Ψ, we get the Euler-Lagrange equations (cf. Lemma (I.1))

(i) $\partial/\partial\kappa \lrcorner \Psi = (f' - \lambda_2)\omega \equiv 0 \bmod N$,

(ii) $\partial/\partial\theta^2 \lrcorner \Psi = -d\lambda_2 - \lambda_1\omega \equiv 0 \bmod N$,

(iii) $\partial/\partial\theta^1 \lrcorner \Psi = -d\lambda_1 - (\kappa f - \kappa^2\lambda_2 - k\lambda_2)\omega \equiv 0 \bmod N$.

The independence condition $\omega_N \neq 0$ and (i) give $\lambda_2 = f'(\kappa)$. Putting this and (ii) into (iii) we get a 2nd order differential equation in κ with leading coefficient $f''(\kappa)$. Thus in order that the Euler-Lagrange equations be trivial, we must have

$$f(\kappa) = a\kappa + b, \qquad a, b \in \mathbf{R}.$$

Putting these back into the Euler-Lagrange equations (i)–(iii) give

(a) if $k \neq 0$, then $a = b = 0$, i.e. $f = 0$;

(b) if $k = 0$, then $b = 0$, i.e. $f = a\kappa$, $a \in \mathbf{R}$.

Next, suppose that the proposition holds for $n - 1$. Similar to the case $n = 0$, we set

$$X = \mathcal{F}(S) \times \mathbf{R}^{n+1}, \quad \text{where } \mathbf{R}^{n+1} \text{ has coordinates } (\kappa, \dot\kappa, \dots, \kappa^{(n)}),$$
$$I = \begin{cases} \theta^i = d\kappa^{(i-1)} - \kappa^{(i)}\omega & (1 \leq i \leq n), \\ \theta^{n+1} = \omega^2 = 0, \\ \theta^{n+2} = \omega_1^2 - \kappa\omega = 0, \end{cases}$$
$$\omega = \omega^1 = ds \neq 0, \qquad \phi = f(\kappa, \dots, \kappa^{(n)})\omega.$$

Again, following the prescription in §I and using the structure equations (2) we compute $\Psi = d(\phi + \lambda_\alpha\theta^\alpha)$ $(\alpha = 1, \dots, n+2)$. Denoting by $\{\partial/\partial\omega, \partial/\partial\kappa, \partial/\partial\kappa^{(i)}, \partial/\partial\theta^{n+1}, \partial/\partial\theta^{n+2}\}$ the dual frame of the coframe $\{\omega, d\kappa, d\kappa^{(i)}, \theta^{n+1}, \theta^{n+2}\}$ $(i = 1, \dots, n)$ of X and contracting the vector fields $\partial/\partial\kappa, \partial/\partial\kappa^{(i)}, \partial/\partial\theta^{n+1}$, and $\partial/\partial\theta^{n+2}$ with Ψ, we get the Euler-Lagrange equations (cf. Lemma (I.1))

(i) $\partial/\partial\kappa^{(n)} \lrcorner \Psi = (f_{\kappa^{(n)}} - \lambda_n)\omega \equiv 0 \bmod N$,

(ii) $\partial/\partial\kappa^{(j)} \lrcorner \Psi = (f_{\kappa^{(j)}} - \lambda_j)\omega - d\lambda_{j+1} \equiv 0 \bmod N$ $(j = 1, \dots, n-1)$,

(iii) $\partial/\partial\kappa \lrcorner \Psi = (f_\kappa - \lambda_{n+2})\omega - d\lambda_1 \equiv 0 \bmod N$,

(iv) $\partial/\partial\theta^{n+2} \lrcorner \Psi = -d\lambda_{n+2} - \lambda_{n+1}\omega \equiv 0 \bmod N$,

(v) $\partial/\partial\theta^{n+1} \lrcorner \Psi = -d\lambda_{n+1} - (\kappa f - \kappa\kappa^{(i)}\lambda_i - \kappa^2\lambda_{n+2} - k\lambda_{n+2})\omega \equiv 0 \bmod N$ $(i = 1, \dots, n)$.

Putting equations (i)–(iv) into (v) we get a $(2n+2)$nd order differential equation in κ with leading coefficient $f_{\kappa^{(n)}\kappa^{(n)}}$. Thus in order that the Euler-Lagrange equations be trivial, we must have

$$f(\kappa, \dot\kappa, \dots, \kappa^{(n)}) = f^0(\kappa, \dot\kappa, \dots, \kappa^{(n-1)}) + \kappa^{(n)} f^1(\kappa, \dot\kappa, \dots, \kappa^{(n-1)})$$

for some functions f^0, f^1. Let $F = F(\kappa, \dot\kappa, \dots, \kappa^{(n-1)})$ be a function satisfying $F_{\kappa^{(n-1)}} = f^1$. Then

$$\frac{dF}{ds} = \kappa^{(n)} f^1 + \sum_{p=0}^{n-2} \kappa^{(p+1)} F_{\kappa^{(p)}}$$

and so

$$\begin{aligned} \oint f\,ds &= \oint f^0\,ds + \oint dF - \oint \sum_{p=0}^{n-2} \kappa^{(p+1)} F_{\kappa^{(p)}}\,ds \\ &= \oint \left[f^0 - \sum_{p=0}^{n-2} \kappa^{(p+1)} F_{\kappa^{(p)}} \right] ds, \end{aligned}$$

where the integrand in the last integral is a function in $(\kappa, \dot{\kappa}, \ldots, \kappa^{(n-1)})$, and the proof is complete by induction. Q.E.D.

Next, consider the loop space $C_I^\infty(S^1, \mathbf{R}^3)$ of regular closed curves in $\mathbf{R}^3$. Let $\gamma \in C_I^\infty(S^1, \mathbf{R}^3)$, and let $\kappa = \kappa(s)$ and $\tau = \tau(s)$ be the curvature and torsion of γ expressed as functions of the arc-length s. It is well known that γ is uniquely determined by $\kappa(s)$ and $\tau(s)$ up to a rigid motion in $\mathbf{R}^3$ and thus functionals on $C_I^\infty(S^1, \mathbf{R}^3)$ are of the form $\Phi(\gamma) = \oint_\gamma f\, ds$, where $f = f(\kappa, \tau, \dot{\kappa}, \dot{\tau}, \ldots)$ (again, the over-dots stand for the derivatives with respect to s). Hence the problem of determining C^∞-invariant functionals on $C_I^\infty(S^1, \mathbf{R}^3)$ reduces to that of determining those functions of the form $f = f(\kappa, \tau, \dot{\kappa}, \dot{\tau}, \ldots)$ with the property that the Euler-Lagrange equations associated to the functional $\Phi(\gamma)$ $(\gamma \in C_I^\infty(S^1, \mathbf{R}^3))$ are trivial.

PROPOSITION (II.2). *There is no nontrivial C^∞-invariant on $C_I^\infty(S^1, \mathbf{R}^3)$, $C_I^\infty(S^1, \mathbf{R}^3)|\kappa \equiv 1$, or $C_I^\infty(S^1, \mathbf{R}^3)|\tau \equiv 1$.*

PROOF. We first consider the space $C_I^\infty(S^1, \mathbf{R}^3)$. It suffices to show that for all $n \geq 0$, the only Lagrangian functions $f = f(\kappa, \tau, \dot{\kappa}, \dot{\tau}, \ldots)$ having trivial Euler-Lagrange equations are exact. We shall use induction on n.

For $n = 0$, i.e. $f = f(\kappa, \tau)$, let $\mathcal{F}(\mathbf{R}^3)$ be the orthonormal frame bundle of $\mathbf{R}^3$. It is well known that $\dim \mathcal{F}(\mathbf{R}^3) = 6$ and there is a coframing $\{\omega^1, \omega^2, \omega^3, \omega_1^2, \omega_1^3, \omega_2^3\}$ satisfying the structure equations [**4, 5**]

$$\begin{cases} d\omega^i = \omega^j \wedge \omega_j^i, \\ d\omega_i^j = \omega_i^k \wedge \omega_k^j, \end{cases} \tag{3}$$

where $1 \leq i,\, j,\, k \leq 3$; $\omega_j^i = -\omega_i^j$. Set

$$X = \mathcal{F}(\mathbf{R}^3) \times \mathbf{R}^2, \quad \text{where } \mathbf{R}^2 \text{ has coordinates } (\kappa, \tau),$$

$$I = \begin{cases} \theta^1 = \omega^2 = 0, \\ \theta^2 = \omega^3 = 0, \\ \theta^3 = \omega_1^3 = 0, \\ \theta^4 = \omega_1^2 - \kappa\omega = 0, \\ \theta^5 = \omega_2^3 - \tau\omega = 0, \end{cases} \qquad \omega = \omega^1 = ds \neq 0, \quad \phi = f(\kappa, \tau)\omega.$$

It is clear that there is a natural (once the orientation is fixed) one-to-one correspondence between curves $\gamma \subset \mathbf{R}^3$ with $\kappa \neq 0$ and $N \in \mathcal{I}(I, \omega)$ in X (of course, the N's are just the Frenet liftings of the γ's), and that our problem reduces to determining the condition for $f(\kappa, \tau)$ so that $(X; I, \omega; \phi)$ has trivial Euler-Lagrange equations.

Following the prescription in §I and using the structure equations (3) we compute $\Psi = d(\phi + \lambda_\alpha \theta^\alpha)$ $(\alpha = 1, \ldots, 5)$. Denoting by $\{\partial/\partial\omega, \partial/\partial\theta^\alpha, \partial/\partial\kappa, \partial/\partial\tau\}$ the dual frame of the coframe $\{\omega, \theta^\alpha, d\kappa, d\tau\}$ of X and contracting the vector fields $\partial/\partial\kappa$, $\partial/\partial\tau$, and $\partial/\partial\theta^\alpha$ with Ψ, we get the Euler-Lagrange equations (cf. Lemma (I.1))

(i) $\partial/\partial\kappa \lrcorner \Psi = (f_\kappa - \lambda_4)\omega \equiv 0 \bmod N$,
(ii) $\partial/\partial\tau \lrcorner \Psi = (f_\tau - \lambda_5)\omega \equiv 0 \bmod N$,
(iii) $\partial/\partial\theta^5 \lrcorner \Psi = -d\lambda_5 - \lambda_3\kappa\omega \equiv 0 \bmod N$,
(iv) $\partial/\partial\theta^4 \lrcorner \Psi = -d\lambda_4 - (\lambda_1 - \lambda_3\tau)\omega \equiv 0 \bmod N$,
(v) $\partial/\partial\theta^3 \lrcorner \Psi = -d\lambda_3 - (\lambda_2 + \lambda_4\tau - \lambda_5\kappa)\omega \equiv 0 \bmod N$,
(vi) $\partial/\partial\theta^2 \lrcorner \Psi = -d\lambda_2 - \lambda_1\tau\omega \equiv 0 \bmod N$,
(vii) $\partial/\partial\theta^1 \lrcorner \Psi = -d\lambda_1 - (\kappa f - \lambda_2\tau - \lambda_4\kappa^2 - \lambda_5\kappa\tau)\omega \equiv 0 \bmod N$.

Putting equations (i)–(v) into (vi) and (vii), we get

$$\begin{cases} f_{\kappa\tau}\kappa^{(3)} + f_{\tau\tau}\tau^{(3)} = (\text{terms of order} \leq 1), \\ (\tau f_{\kappa\tau} + \kappa f_{\kappa\kappa})\ddot{\kappa} + (\tau f_{\tau\tau} + \kappa f_{\kappa\tau})\ddot{\tau} = (\text{terms of order} \leq 1). \end{cases}$$

Thus in order that the Euler-Lagrange equations associated to $(X; I, \omega; \phi)$ be trivial, we must have $f_{\kappa\kappa} = f_{\kappa\tau} = f_{\tau\tau} = 0$, i.e.

$$f(\kappa, \tau) = a\kappa + b\tau + c, \qquad a, b, c \in \mathbf{R}.$$

But then equations (i)–(vii) yield $f = 0$.

Next, suppose the proposition holds for $n - 1$. We set

$$X = \mathcal{F}(\mathbf{R}^3) \times \mathbf{R}^{2n+2}, \quad \text{where } \mathbf{R}^{2n+2} \text{ has coordinates} (\kappa, \tau, \dot{\kappa}, \dot{\tau}, \ldots, \kappa^{(n)}, \tau^{(n)}),$$

$$I = \begin{cases} \theta^i = d\kappa^{(i-1)} - \kappa^{(i)}\omega = 0, & 1 \leq i \leq n, \\ \theta^{n+i} = d\tau^{(i-1)} - \tau^{(i)}\omega = 0, & 1 \leq i \leq n, \\ \theta^{2n+1} = \omega^2 = 0, \\ \theta^{2n+2} = \omega^3 = 0, \\ \theta^{2n+3} = \omega_1^3 = 0, \\ \theta^{2n+4} = \omega_1^2 - \kappa\omega = 0, \\ \theta^{2n+5} = \omega_2^3 - \tau\omega = 0, \end{cases}$$

$$\omega = \omega^1 = ds \neq 0, \qquad \phi = f(\kappa, \tau, \dot{\kappa}, \dot{\tau}, \ldots, \kappa^{(n)}, \tau^{(n)})\omega.$$

By computing $\Psi = d(\phi + \lambda_\alpha\theta^\alpha)$ $(\alpha = 1, \ldots, 2n + 5)$ and contracting all the vector fields on X with Ψ, we get the Euler-Lagrange equations (1). Using the same technique as that in Proposition (II.1), we can show that the triviality of the Euler-Lagrange equations implies that

$$f_{\kappa^{(n)}\kappa^{(n)}} = f_{\kappa^{(n)}\tau^{(n)}} = f_{\tau^{(n)}\tau^{(n)}} = 0.$$

Hence we have

$$f = f^0 + \kappa^{(n)} f^1 + \tau^{(n)} f^2,$$

where f^0, f^1, and f^2 are functions in $(\kappa, \tau, \ldots, \kappa^{(n-1)}, \tau^{(n-1)})$. Let

$$F(\kappa, \tau, \ldots, \kappa^{(n-1)}, \tau^{(n-1)})$$

be a function satisfying $F_{\kappa^{(n-1)}} = f^1$; then

$$\frac{dF}{ds} = \kappa^{(n)} f^1 + \tau^{(n)} F_{\tau^{(n-1)}} + \sum_{p=0}^{n-2} (\kappa^{(p+1)} F_{\kappa^{(p)}} + \tau^{(p+1)} F_{\tau^{(p)}})$$

and thus

$$\begin{aligned} \oint f\, ds = & \oint f^0\, ds + \oint dF + \oint \tau^{(n)}(f^2 - F_{\tau^{(n-1)}})\, ds \\ & - \oint \sum_{p=0}^{n-2} (\kappa^{(p+1)} F_{\kappa^{(p)}} + \tau^{(p+1)} F_{\tau^{(p)}})\, ds \\ = & \oint (g + \tau^{(n)} h)\, ds, \end{aligned}$$

where g and h are functions in $(\kappa, \tau, \ldots, \kappa^{(n-1)}, \tau^{(n-1)})$. Therefore we may just assume that $f = g + \tau^{(n)} h$ and by putting this f back into the Euler-Lagrange equations and repeating the above process, we get $h_{\kappa^{(n-1)}} = 0$, i.e.

$$h = h(\kappa, \tau, \ldots, \kappa^{(n-2)}, \tau^{(n-2)}, \tau^{(n-1)}).$$

Let $H(\kappa, \tau, \ldots, \kappa^{(n-2)}, \tau^{(n-2)}, \tau^{(n-1)})$ be a function satisfying $H_{\tau^{(n-1)}} = h$; then

$$\frac{dH}{ds} = \tau^{(n)} h + \sum_{p=0}^{n-2} (\kappa^{(p+1)} H_{\kappa^{(p)}} + \tau^{(p+1)} H_{\tau^{(p)}})$$

and so

$$\begin{aligned}\oint f\, ds &= \oint g\, ds + \oint dH - \oint \sum_{p=0}^{n-2} (\kappa^{(p+1)} H_{\kappa^{(p)}} + \tau^{(p+1)} H_{\tau^{(p)}})\, ds \\ &= \oint l(\kappa, \tau, \ldots, \kappa^{(n-1)}, \tau^{(n-1)})\, ds.\end{aligned}$$

Now by the induction hypothesis, we see that f is exact. This completes the proof of the proposition for the space $C_I^\infty(S^1, \mathbf{R}^3)$.

Now for the spaces $C_I^\infty(S^1, \mathbf{R}^3)|\kappa \equiv 1$ and $C_I^\infty(S^1, \mathbf{R}^3)|\tau \equiv 1$, we see that they are just constrained problems posed in the manifold $X = \mathcal{F}(\mathbf{R}^3) \times \mathbf{R}^{2n+2}$. Letting $X_1 = \{\kappa \equiv 1\} \subset X$, $X_2 = \{\tau \equiv 1\} \subset X$, $I_i = I|X_i$, $f_i = f|X_i$, we proceed to find the conditions for f_i so that the Euler-Lagrange equations associated to the problems $(X_i; I_i, \omega; \phi_i = f_i\omega)$ are trivial. Following exactly the same arguments as that for X it is not hard to conclude that f_i is exact for $i = 1, 2$ and thus completes the proof of the proposition. Q.E.D.

References

1. R. Bryant, S. S. Chern, and P. Griffiths, *Exterior differential systems*, Proc. Beijing Sympos. on Differential Geometry and Differential Equations, 1980.
2. E. Cartan, *Les systèmes différentielles extérieurs et leurs applications géométriques*, Hermann, Paris, 1945.
3. W. S. Cheung, *Higher order conservation laws and a higher order Noether's theorem*, preprint.
4. P. Griffiths, *Exterior differential systems and the calculus of variations*, Birkhäuser, Boston, Mass., 1983.
5. S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

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