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POLYNOMIALS OF 2-CABLE-LIKE LINKS

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ABSTRACT. Morton and Short [**MS**] have established experimentally that two knots K_1 and K_2 may have the same 2-variable polynomial $P(l,m)$ (see [**FYHLMO**], [**LM**]) while 2-cables on K_1 and K_2 can be distinguished by P. We prove here that if K_1 and K_2 are a mutant pair, then their 2-cables and doubles (and other satellites which are 2-stranded on the boundary of the mutating tangle) cannot be distinguished by P. Similar results are true for the unoriented knot polynomial Q and its oriented two-variable counterpart F (see [**BLM**], [**K**]). The results are false if K_1, K_2 are links of more than one component.

1. Preliminaries. There exists for each oriented link L in the 3-sphere a 2-variable Laurent polynomial $P_L(l,m) \in \mathbf{Z}[l^{\pm 1}, m^{\pm 1}]$ defined uniquely by the following:

(i) $P_U = 1$ for the unknot U,

(ii) $lP_{L_+} + l^{-1}P_{L_-} + mP_{L_0} = 0$,

where L_+, L_-, and L_0 are identical outside a ball and inside are as shown in Figure 1(a) (see [**FYHLMO**] and [**LM**]).

Similarly, we may uniquely assign to each nonoriented link L a 1-variable Laurent polynomial $Q_L(x) \in \mathbf{Z}[x^{\pm 1}]$ such that:

(iii) $Q_U = 1$ for the unknot U,

(iv) $(Q_{L_+} + Q_{L_-}) - x(Q_{L_0} + Q_{L_\infty}) = 0$,

where L_+, L_-, L_0, and L_∞ are identical outside a ball and inside it are as shown in Figure 1(b) (see [**BLM**]).

Suppose the knot K is made up of tangles R, S, as in Figure 2(a). Any knot obtained by rotating R about one of the three axes shown is a *mutant* of K. We denote the images of R under these transformations by $\rho R, \sigma R, \tau R$ and the corresponding mutants of K by $\rho K, \sigma K, \tau K$ respectively. Recall that $P_K = P_{\mu K}$ and $Q_K = Q_{\mu K}$ for any mutation μ.

2. The results.

THEOREM 1. *Let the oriented knot K_2 be obtained from an oriented knot K_1 by mutation of the tangle R. Let K_1', K_2' be $(2,n)$-cables about K_1, K_2. Then K_1' and K_2' share the same 2-variable polynomial $P(l,m)$.*

We prove Theorem 1 in §3. In §4 we prove the following.

THEOREM 2. *Let K_1, K_2 be as in Theorem 1 and K_1', K_2' be doubles (or $(2,2n)$-cables in which the orientation of one component has been reversed) of K_1, K_2. Then K_1' and K_2' share the same 2-variable polynomial $P(l,m)$.*

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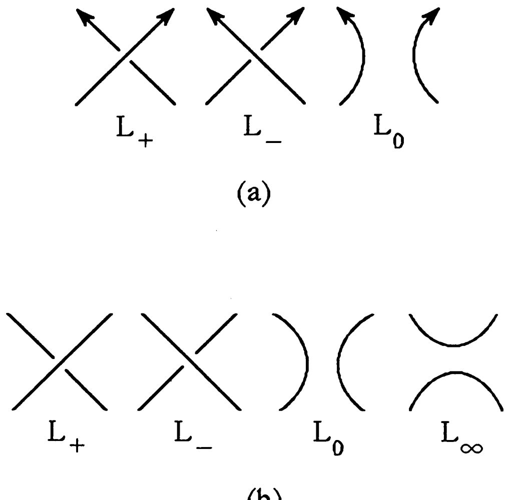


FIGURE 1

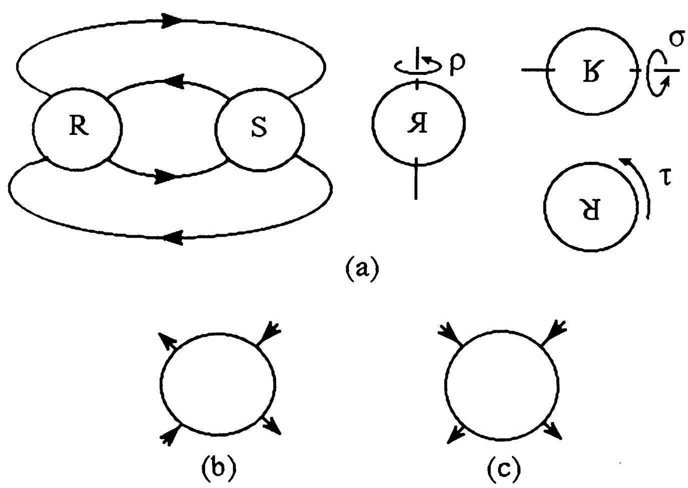


FIGURE 2

In §5 we show a similar proof for the following theorem.

THEOREM 3. *Let the unoriented knot K_2 be obtained from the unoriented knot K_1 by mutation of a tangle R. Let K_1', K_2' be $(2,n)$-cables or doubles about K_1, K_2. Then K_1' and K_2' share the same* 1*-variable knot polynomial $Q(x)$.*

Essentially the same proof gives the result for Kaufmann's 2-variable polynomial $F(a,x)$ (see [**K**]).

It follows from these results that *any* satellite of a knot K which looks like a cable or double (i.e. is 2-stranded) on the boundary of a tangle R can be 'mutated' at that tangle without changing the value of P, Q, or F. Such links are skein-generated (see [**LM**]) by doubles and 2-cables about K.

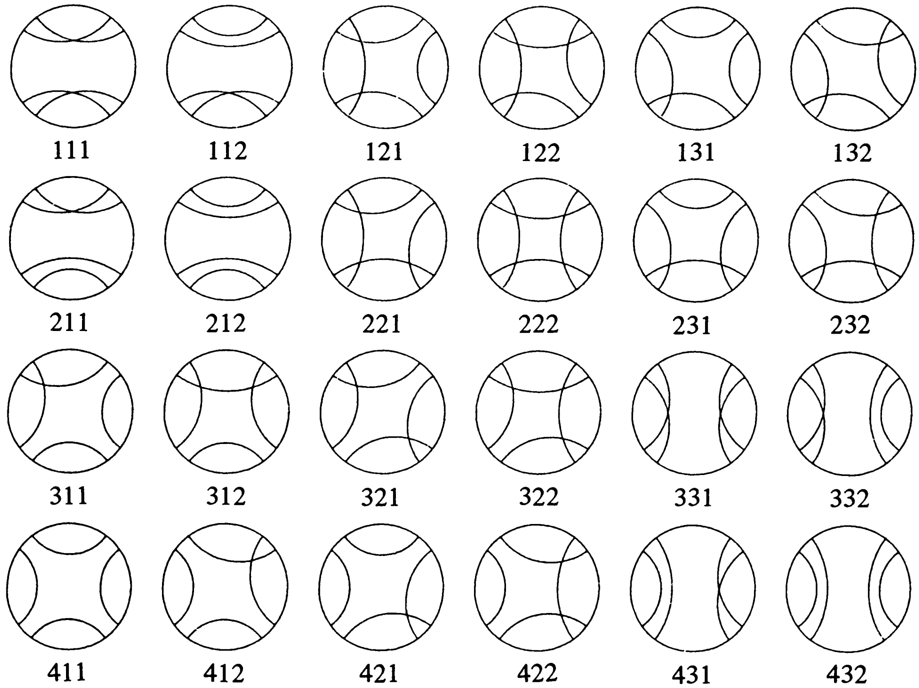
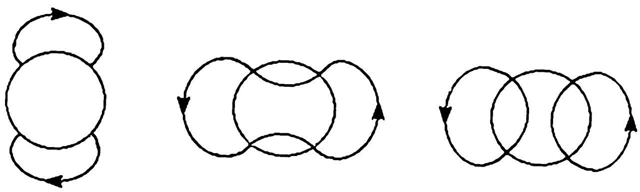


FIGURE 3

3. Polynomials of 2-cables about mutants. Let the oriented knot K_2 be obtained from an oriented knot K_1 by mutation of the tangle R. We suppose without loss of generality that in the chosen knot projection the endpoints of each arc in R are adjacent to each other in the boundary of R. (So R is as shown in Figure 2(b), not as in Figure 2(c).) Since we are not dealing with links of more than one component, this condition is automatically true for the tangle constituting the exterior of R. Note that the mutations ρ and σ involve a change of orientation for the arcs inside R. Let K_1' be a $(2,n)$-cable about K_1; the tangle R becomes R', and we need to consider the effect of the rotations ρ, σ, τ on R'.

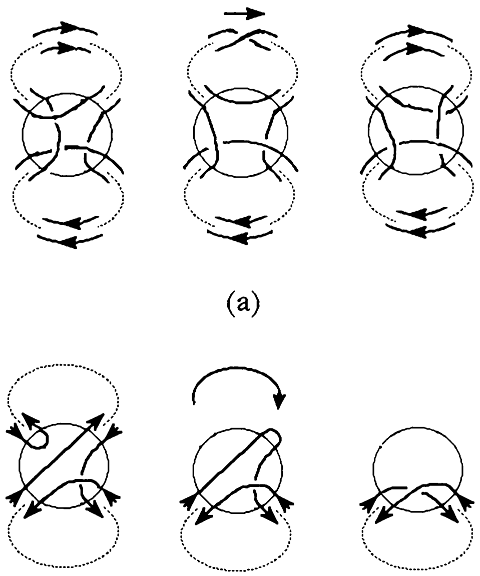
By repeated application of the polynomial identity (ii) we may express the 2-variable polynomial of K_1' in terms of those of the 24 links which are identical with K_1' outside the tangle R' and which inside it take the forms shown in Figure 3. In the terminology of [**LM**], the 24 diagrams of Figure 3 skein-generate the room inhabited by R', and the links obtained by substituting these tangles for R' generate K_1'. This statement remains true regardless of which signs we choose to give the crossings in Figure 3.

We consider the rotation ρ. Suppose that the knot K_1 with which we started was joined up schematically as in Figure 4(a). That is, the arc leaving R at the top left first re-enters R at top right. Then it is not difficult to see that each of the 24 generators is invariant under the rotation: the cases 111, 112, 211, 212, 222, 231, 312, 331, 411, and 432 can all be taken to be symmetric under ρ by appropriate choice of crossing signs and the other cases can be disposed of by observing that the external connections of the knot allow one to pass a half-twist from top left of the tangle to top right and vice versa, or from bottom left to bottom right and



(a) (b) (c)

FIGURE 4



(a)

(b)

FIGURE 5

vice versa. In Figure 5(a) we show e.g. that 221 represents the same knot before application of ρ as after. The other cases proceed in exactly the same way. It follows then that in the case where K_1 is joined up in this way, its $(2, n)$-cable has the same P-polynomial as the $(2, n)$-cable of the ρ-mutant K_2 of K_1.

Now suppose K_1 is joined up as in Figure 4(b), outside the tangle R. It follows that inside R it must be joined up as shown, schematically, for we assume that K_1 is a knot, and were it joined up as in Figure 4(c) it would have at least two components. Note now that ρ-rotation of the tangle R is equivalent to ρ-rotation of the exterior of R, and by inverting in the boundary of R we obtain the situation considered above. Hence we find that in this case also, the P-polynomial of the $(2, n)$-cable of K_1 is identical to that of the $(2, n)$-cable of the ρ-mutant K_2 of K_1.

Now by symmetry it is clear that the same argument can be applied to the mutation σ. Finally we note that $\tau = \sigma\rho$ and so the result is also true for τ-mutants.

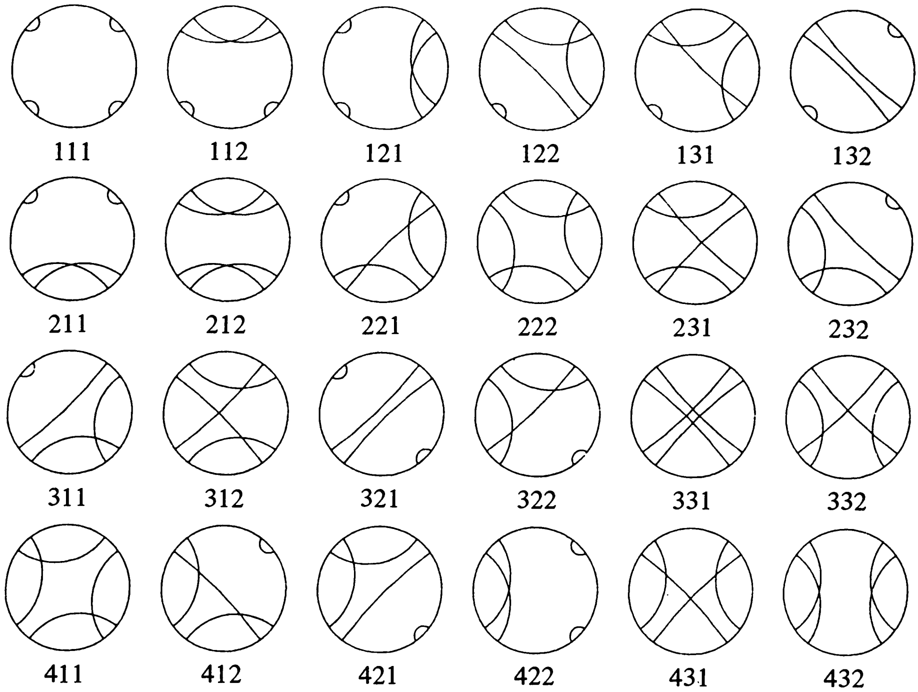
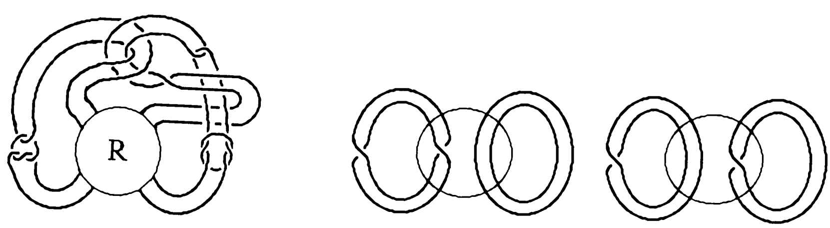


FIGURE 6



(a) (b)

FIGURE 7

4. Doubles and other satellites. We prove Theorem 2 in a very similar way, observing that in this case the 24 diagrams in Figure 6 skein-generate the room inhabited by the doubled tangle R_1'.

As before, we may choose the crossings as we like. There is one extra trick required to obtain this result. We illustrate it for the diagram 311 of Figure 6. We allow ourselves to 'retract' a finger along the path followed by K_1 outside R (see Figure 5(b)). This transforms the diagram (without changing the knot) into a form which makes it obvious that its polynomial will be unchanged by ρ-mutation. This procedure, coupled with the method used in §3 to prove Theorem 1, is sufficient to obtain the required result.

As an immediate corollary we find that any satellite of K_1 which is generated in the skein of links by cables or doubles of K_1 will have a 2-variable polynomial which is invariant under mutation of K_1. An example of such a link is shown in Figure 7(a).

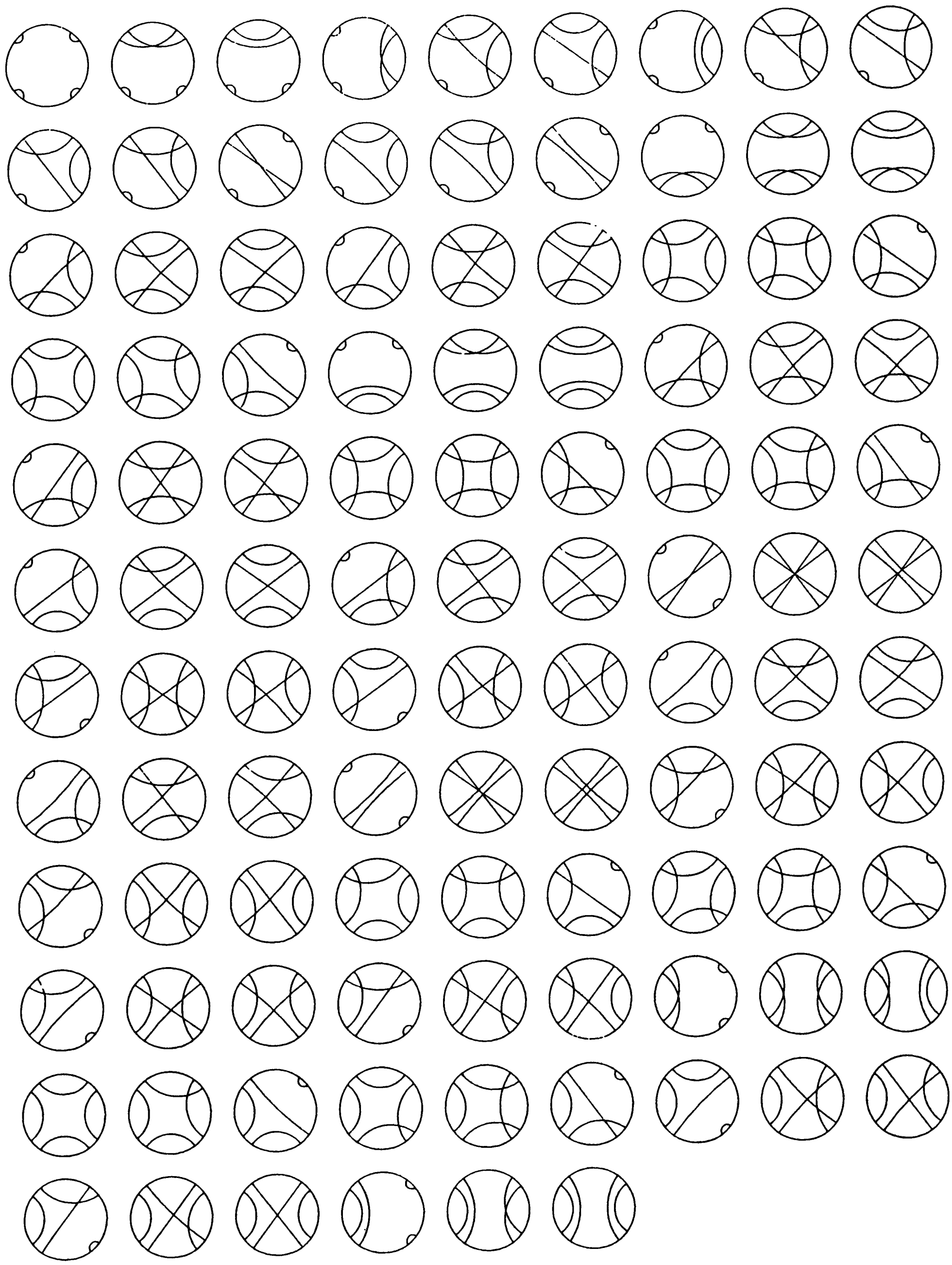


FIGURE 8

Note that the hypothesis that K_1 be a knot (that is, a link of one component) is strictly necessary. For otherwise we could conclude that a 2-component simple link had the same polynomial as the null link, which is false (Figure 7(b)).

5. The Q-polynomial and concluding remarks. The proof of Theorem 3 proceeds in exactly the same way as those of Theorems 1 and 2. However, since we are now dealing with unoriented links it is necessary to consider a larger set of (105) generators for the skein we are considering (see Figure 8). No new tricks are required: passing half-twists along sections external to R and retracting fingers can cope with all of the cases. A little care has to be exercised over the choice of crossing signs in the diagrams, though.

The same proof shows the result to be true for Kaufmann's F-polynomial (see [**K**]).

Once again the result fails for 2-cables and doubles on links of more than one component, with the same counterexample as before. This is in some sense the canonical counterexample since the main restriction to constructing a proof of a more general (false) result is the impossibility of passing a half-twist from the cable on one component of a link to the cable on another.

We comment that the restriction on R which was first invoked in the proof of Theorem 1 does not actually restrict our results, and proofs for other tangles follow from the arguments given here.

In conclusion we warn the reader that examination of a suitable set of 720 generators for the case of 3-cables on mutants has made it appear unlikely that our results here generalize to 3-stranded satellites!

REFERENCES

[**BLM**] R. D. Brandt, W. B. R. Lickorish and K. C. Millett, *A polynomial invariant for unoriented knots and links*, Invent. Math. **84** (1986), 563–573.

[**FYHLMO**] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millet and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 239–246.

[**K**] L. H. Kaufmann, *An invariant of regular isotopy* (to appear).

[**LM**] W. B. R. Lickorish and K. Millett, *A polynomial invariant of oriented links*, Topology (to appear).

[**MS**] H. R. Morton and H. B. Short, *The 2-variable polynomial of cable knots* (to appear).

[**R**] D. Rolfsen, *Knots and links*, Publish or Perish, Berkeley, Calif., 1976.

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