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LEVEL SETS FOR FUNCTIONS CONVEX IN ONE DIRECTION

JOHNNY E. BROWN

ABSTRACT. Goodman and Saff conjectured that if f is convex in the direction of the imaginary axis then so are the functions $\frac{1}{r}f(rz)$ for all $0 < r < \sqrt{2} - 1$, i.e., the level sets $f(|z| < r)$ are convex in the direction of the imaginary axis for $0 < r < \sqrt{2} - 1$. A weak form of this conjecture is proved and a question of Brannan is answered negatively.

Let $U_r = \{z : |z| < r\}$ and let S denote the class of all functions $f(z) = z + a_2z^2 + \cdots$ analytic and univalent in $U = U_1$. For $f \in S$ there are several geometric properties possessed by $f(U)$ that are inherited by its level sets $G_r \equiv f(U_r)$ for all $0 < r < 1$. For example if $f(U)$ is either convex, starlike, or close-to-convex, then so are its level sets G_r for all $0 < r < 1$.

An analytic function f is said to be convex in the direction of a line L_θ: $te^{i\theta}$ $(-\infty < t < \infty)$ if the intersection of $f(U)$ with each line parallel to L_θ is either a connected set or empty. Let CIA denote those functions f for which $f(U)$ is convex in the direction of the imaginary axis with $f(0) = 0$ and $f'(0) = 1$. Since CIA functions are close-to-convex, they are univalent. It came as a bit of a surprise when Hengartner and Schober [**4**] constructed an example where $f \in CIA$ but the corresponding level sets G_r were not convex in the direction of the imaginary axis for all r sufficiently close to 1. A more quantitative result was obtained by Goodman and Saff [**3**]. They were able to prove that for each $\sqrt{2} - 1 < r < 1$ there exists an $f \in CIA$ for which $\frac{1}{r}f(rz) \notin CIA$. Hence they conjectured that if $f \in CIA$ then $\frac{1}{r}f(rz) \in CIA$ for all $0 < r < \sqrt{2} - 1$, i.e., the level sets G_r are convex in the direction of the imaginary axis for $0 < r < \sqrt{2} - 1$.

In [**1**, Problem 6.53] Brannan asked whether or not: If $f \in CIA$ and $\frac{1}{r_0}f(r_0z) \notin CIA$ for some $0 < r_0 < 1$, does this imply that $\frac{1}{r}f(rz) \notin CIA$ for all $r_0 < r < 1$? This question was motivated by the example constructed by Hengartner and Schober [**4**]. In this note we prove a weaker form of the Goodman-Saff conjecture and answer Brannan's question. Our main result is the following theorem.

THEOREM. *If $f \in CIA$ then there exists a set $I \subseteq [0, \pi]$ of positive measure such that $\frac{1}{r}f(rz)$ is convex in the directions L_θ: $te^{i\theta}$ $(-\infty < t < \infty)$ for all $0 < r < \sqrt{2} - 1$ and all $\theta \in I$.*

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The proof of this theorem depends on a representation formula for CIA functions and a result relating the total variation of the argument along a curve to the number of times the curve intersects lines through the origin.

LEMMA A (ROYSTER AND ZIEGLER [**8**]). *A function f belongs to CIA if and only if*

$$\operatorname{Re}\left\{\frac{-izf'(z)}{h_\nu(e^{-i\mu}z)}\right\} > 0, \qquad z \in U,$$

for some $0 \leqslant \mu, \nu \leqslant \pi$ *where* $h_\nu(z) = z/[1 - (2\cos\nu)z + z^2]$.

A function f analytic in U is said to be starlike in the direction L_θ: $te^{i\theta}$ $(-\infty < t < \infty)$ if the intersection of $f(U)$ with L_θ is a single segment, half-line, or L_θ. The following result gives a relation between these functions and functions convex in one direction. Our formulation follows easily from the results in [**7**].

LEMMA B (ROBERTSON [**7**]). *If* $zf'(z)$ *is starlike in the direction* L_θ: $te^{i\theta}$ $(-\infty < t < \infty)$ *for* $|z| < r$, *then* $f(z)$ *is convex in the direction* $L_{\theta+\pi/2}$ *for* $|z| < r$.

Finally we need a counting result given in [**9**] and given as a problem in [**2**, p. 215].

LEMMA C. *If* $\phi(z)$ *is analytic in* $|z| \leqslant r$ *and* $0 \notin \phi(|z| = r)$, *then*

$$\int_0^{2\pi}\left|\operatorname{Re}\left\{\frac{z\phi'(z)}{\phi(z)}\right\}\right| d\theta = \int_0^{\pi} n(\psi)\,d\psi, \qquad |z| = r,$$

where $n(\psi)$ *is the number of times the line* $te^{i\psi}$ $(-\infty < t < \infty)$ *intersects the curve* $\phi(|z| = r)$.

Although this result is geometrically obvious, for completeness sake we give a proof using the Banach indicatrix.

PROOF OF LEMMA C. By replacing $\phi(z)$ by $\phi(e^{i\alpha}z)$, if necessary, we may assume $\phi(r) > 0$. Let $0 = \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_n = 2\pi$ be all those angles such that $\phi(re^{i\theta_k}) > 0$ and $\operatorname{Arg}\phi(re^{i\theta})$ varies continuously from 0 to 2π for $\theta_k \leqslant \theta \leqslant \theta_{k+1}$ for each $1 \leqslant k \leqslant n-1$. Note that if the curve $\phi(|z| = r)$ winds around the origin more than once, then $n \geqslant 3$. For each interval $\theta_k \leqslant \theta \leqslant \theta_{k+1}$, $1 \leqslant k \leqslant n-1$, let $H_k(\theta) \equiv \operatorname{Arg}\phi(re^{i\theta})$. Hence $0 \leqslant H_k(\theta) \leqslant 2\pi$. Let $m_k(\psi)$ be equal to the number of roots of the equation $H_k(\theta) = \psi$, $0 \leqslant \psi \leqslant 2\pi$. The function m_k is the Banach indicatrix of the function H_k and thus

$$\int_0^{2\pi} m_k(\psi)\,d\psi = \int_{\theta_k}^{\theta_{k+1}} |dH_k(\theta)|, \qquad 1 \leqslant k \leqslant n-1$$

(see Natanson [**6**, p. 225] for example). Now since $|dH_k(\theta)| = |\operatorname{Re}\{z\phi'(z)/\phi(z)\}|\,d\theta$, it follows that

$$\int_0^{2\pi} m(\psi)\,d\psi = \int_0^{2\pi}\left|\operatorname{Re}\left\{\frac{z\phi'(z)}{\phi(z)}\right\}\right| d\theta,$$

where $m(\psi) = m_1(\psi) + m_2(\psi) + \cdots + m_{n-1}(\psi)$ is the number of times the curve $\phi(|z| = r)$ intersects the ray $te^{i\psi}$ $(0 < t < \infty)$ counting multiplities. Finally observe that

$$\int_0^{2\pi} m(\psi)\,d\psi = \int_0^{\pi} m(\psi) + m(\psi + \pi)\,d\psi = \int_0^{\pi} n(\psi)\,d\psi$$

and so the proof of the lemma is complete.

If $\phi(z)$ is analytic and vanishes only for $z = 0$ in $|z| \leqslant r$ then we assert that $n(\psi)$ is a positive even integer function except for finitely many ψ. To see this, suppose that $n(\psi)$ is odd for infinitely many $|z_k| = r$. Since $\phi(|z| = r)$ is an analytic Jordan curve surrounding the origin it follows that $\operatorname{Re}\{z_k\phi'(z_k)/\phi(z_k)\} = 0$. The function $\omega(z) = z\phi'(z)/\phi(z)$ is analytic in $|z| \leqslant r$ and the curve $\omega(|z| = r)$ meets the imaginary axis an infinite number of times. Hence we can conclude that $\omega(z) \equiv i\lambda$ for some $\lambda \in \mathbf{R}$, but since $\omega(0) = 1$ a contradiction is reached. The assertion is proved.

PROOF OF THEOREM. Fix $0 < r < \sqrt{2} - 1$. Observe first that

$$Q(\zeta) \equiv \operatorname{Re}\left\{\frac{\zeta h_\nu'(\zeta)}{h_\nu(\zeta)}\right\} = \operatorname{Re}\left\{\frac{1 - \zeta^2}{1 - (2\cos\nu)\zeta + \zeta^2}\right\},$$

where h_ν is given in Lemma A, is harmonic in U, $Q(0) = 1$, and $Q(\zeta) > 0$. If $p(z)$ is analytic in U, $p(0) = 1$, and $\operatorname{Re} p(z) > 0$, then the following estimate was given by Libera [5]:

$$\left|\frac{zp'(z)\cdot}{p(z) + i\beta}\right| \leqslant \frac{2r}{1 - r^2}, \qquad |z| = r, \tag{1}$$

for any $\beta \in \mathbf{R}$.

Let $f \in CIA$. Then by Lemma A we get

$$zf'(z) = h_\nu(e^{-i\mu}z)[\cos\mu + ip(z)\sin\mu] \tag{2}$$

for some $0 \leqslant \mu,\ \nu \leqslant \pi$ and some function p analytic in U with $p(0) = 1$, $\operatorname{Re} p > 0$. If $\phi(z) = zf'(z)$ then from (2) we obtain

$$\frac{z\phi'(z)}{\phi(z)} = \left\{\frac{(e^{-i\mu}z)h_\nu'(e^{i\mu}z)}{h_\nu(e^{-i\mu}z)}\right\} + \left\{\frac{zp'(z)}{p(z) - i\cot\mu}\right\} \tag{3}$$

(if $\mu = 0$ or π the last term is not present). It follows from (3) and (1) that

$$\int_0^{2\pi}\left|\operatorname{Re}\frac{z\phi'(z)}{\phi(z)}\right|d\theta \leqslant \int_0^{2\pi}\operatorname{Re}\left\{\frac{(e^{-i\mu}z)h_\nu'(e^{-\mu}z)}{h_\nu(e^{-i\mu}z)}\right\}d\theta + \int_0^{2\pi}\left|\frac{zp'(z)}{p(z) - i\cot\mu}\right|d\theta$$

$$\leqslant 2\pi + 2\pi\left(\frac{2r}{1 - r^2}\right) = 2\pi\left(\frac{1 + 2r - r^2}{1 - r^2}\right),$$

where $|z| = r$. By hypothesis, $r < \sqrt{2} - 1$ and hence

$$\int_0^{2\pi}\left|\operatorname{Re}\left\{\frac{z\phi'(z)}{\phi(z)}\right\}\right|d\theta < 4\pi.$$

Now from Lemma C we get

$$\int_0^{\pi} n(\psi)\, d\psi < 4\pi. \tag{4}$$

As pointed out earlier, since $\phi(z) = zf'(z)$ is analytic in $|z| \leqslant r$ and vanishes only at $z = 0$, $n(\psi) \geqslant 2$ and $n(\psi)$ is a positive even integer function except for a finite set of ψ, say $E_0 = \{\psi_1, \psi_2, \dots, \psi_n\}$. The inequality (4) thus gives $n(\psi) < 4$ for all $\psi \in J$ for some set $J \subseteq [0, \pi]$ with positive measure. Hence $n(\psi^*) = 2$ for all $\psi^* \in J \setminus E_0 \equiv I^*$ and so the function $\phi(z) = zf'(z)$ is starlike in the directions L_{ψ^*} for all $\psi^* \in I^*$. We now apply Lemma B to conclude that for each such direction L_{ψ^*}, the function $f(z)$ is convex in the direction L_ψ, where $\psi = \psi^* + \pi/2$ for $|z| \leqslant r$. This completes the proof of the theorem.

We now turn to the question of Brannan. Specifically we find a function $F \in CIA$ with $\frac{1}{r_0}F(r_0 z) \notin CIA$ but $\frac{1}{r_1}F(r_1 z) \in CIA$ for some $r_0 < r_1 < 1$. The function

$$F(z) = \frac{z - Az^2}{(1 - Bz)^2},$$

where $A = e^{2i\alpha}\cos\alpha$ and $B = e^{i\alpha}$, belongs to CIA and maps U onto the exterior of a vertical slit lying along the line $\operatorname{Re} w = -\cos\alpha/2$ (see [3]). If $(\partial/\partial\theta)\operatorname{Re} F(re^{i\theta})$ has exactly two sign changes in $[0, 2\pi)$, then $\frac{1}{r}F(rz) \in CIA$; while if it has four sign changes then $\frac{1}{r}F(rz) \notin CIA$. Following [3] we see that by replacing z by $ze^{-i\alpha}$, the number of sign changes of $(\partial/\partial\theta)\operatorname{Re} F(re^{i\theta})$ is the same as the number of sign changes of $Q(\theta)$ in $[0, 2\pi)$, where

$$Q(\theta) \equiv (1 + r^2)\sin(\theta - \alpha) + r[3\sin\alpha - \sin(2\theta + \alpha)]. \tag{5}$$

In what follows, let $\alpha = 2.6$. Suppose first that $r = r_0 = 0.5$. Then since $Q(0) = Q(2\pi) = -0.1288\dots$, $Q(1) = 0.0206\dots$, $Q(2) = -0.0883\dots$, and $Q(\pi) = 1.1598\dots$ we can conclude that $\frac{1}{r_0}F(r_0 z) \notin CIA$.

Suppose next that $r = r_1 = 0.7$. Table 1 contains Q and Q' correct to six and three decimal places, respectively. Note that since Q and Q' have at most four zeros in $[0, 2\pi)$, we see that Q must have a zero in the interval $(0, 1)$, and in $(4.2, 2\pi)$ and possibly in the interval $(2.05, 2.06)$. It is easy to check that for $2.05 < \theta < 2.06$ we get

$$Q(\theta) > (1.49)(-0.523) + 0.7[3(0.515) - 0.424] > 0.005.$$

Hence Q has exactly two zeros in $[0, 2\pi)$ and so $\frac{1}{r_1}F(r_1 z) \in CIA$ and $r_1 > r_0$.

TABLE 1

θ	$Q(\theta)$	$Q'(\theta)$
0	−0.046391	−0.077
1	0.288771	0.113
2.05	0.020353	−0.009
2.06	0.020351	0.009
4.2	3.271910	−0.049
2π	−0.046391	−0.077

Finally, it should be pointed out that our proof of the theorem cannot yield the Goodman-Saff conjecture directly as it is not sensitive to direction. We have proved the weaker conjecture that if f is convex in one direction then $\frac{1}{r}f(rz)$ is convex in one direction for all $0 < r < \sqrt{2} - 1$. If we could rotate *CIA* functions then the Goodman-Saff conjecture would follow immediately. The classes of convex, starlike, and close-to-convex functions are rotationally invariant, but the class of *CIA* functions clearly is not. Despite this drawback, we believe the Goodman-Saff conjecture is true.

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Department of Mathematics, Purdue University, West Lafayette, Indiana 47907