PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 3, July 1987

AN UNCONDITIONAL RESULT ABOUT GROTHENDIECK SPACES

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ABSTRACT. It is shown that if X is a nonreflexive Banach spach such that every weak* convergent sequence in X^* is also weakly convergent, then X^* has a subspace isometric to $L_1(\{0,1\}^{\omega_1})$

It was shown in a recent article [**4**] that if the cardinal $\mathfrak{p}$ is greater than ω_1 (which is the case if we assume Martin's Axiom and the negation of the Continuum Hypothesis), then every nonreflexive Grothendieck space contains a subspace isomorphic to $l_1(\mathfrak{p})$. It is known that this result is false if we assume the Continuum Hypothesis [**6**]. This paper presents a proof without special axioms that if X is a nonreflexive Grothendieck space, then X^* has a subspace isometric to $L_1(\{0,1\}^{\mathfrak{p}})$. In fact, the hypotheses on X in Theorem 1 are somewhat weaker, being those of the Hagler-Johnson theorem (Theorem 1(a) of [**3**]).

Notation and terminology for Banach spaces are standard. A Banach space X is a *Grothendieck space* if every weak* convergent sequence in X^* is also weakly convergent. For a set I, $L_1(\{0,1\}^I)$ denotes the L_1-space for the usual measure on $\{0,1\}^I$. We can regard this either as the infinite product of measures $\frac{1}{2}(\delta_0+\delta_1)$ on each coordinate, or as Haar measure on the compact group $\{0,1\}^I$.

Cardinals are as usual identified with initial ordinals; ω is the first infinite cardinal, and is also the set of natural numbers; ω_1 is the first uncountable cardinal. The continuum 2^ω is denoted by $\mathfrak{c}$ and the cardinal of a set I by $\#I$.

The cardinal $\mathfrak{p}$ is the smallest cardinal such that there exists a family $(M_\xi)_{\xi\in\mathfrak{p}}$ of subsets of ω such that

(a) $\bigcap_{\xi\in Q} M_\xi$ is infinite for all finite $Q\subset\mathfrak{p}$;

(b) there is no infinite M with $M\setminus M_\xi$ finite for all $\xi\in\mathfrak{p}$.

Rephrased succinctly, one can diagonalize fewer than $\mathfrak{p}$ compatible infinite subsets of ω. An account of $\mathfrak{p}$ and its relation to other cardinals can be found in [**1**] or [**2**]. It is known, for instance, that $\mathfrak{p}$ is a regular cardinal with $\omega_1\le\mathfrak{p}\le\mathfrak{c}$.

We are now in a position to state our theorem.

THEOREM 1. *Let X be a Banach space and assume that in X^* there is an infinite-dimensional subspace Y such that every weak* convergent sequence in Y is also norm convergent. Then X^* has a subspace isometric to $L_1(\{0,1\}^{\mathfrak{p}})$.*

COROLLARY. *If X is a nonreflexive Grothendieck space, then X^* has a subspace isometric to $L_1(\{0,1\}^{\mathfrak{p}})$.*

PROOF OF THE COROLLARY. Since X^* is not reflexive it contains a sequence with no weakly convergent subsequence. Since every weak Cauchy sequence is

Received by the editors May 12, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46B20; Secondary 46E30.

weak* convergent, and hence weakly convergent, we see that our sequence has no weak Cauchy subsequence. So by Rosenthal's l_1 theorem [**5**], X^* has a subspace isomorphic to l_1. By the Schur property of l_1 and the Grothendieck property of X, this subspace will do as Y in the theorem.

The starting point for the proof of the theorem is the following lemma, taken from [**3**]. Recall that g is said to be an l_1-normalized linear combination of f_n $(n \in \omega)$ if there exist a finite subset $B \subset \omega$ and real numbers α_n $(n \in B)$ such that

$$g = \sum_{n \in B} \alpha_n f_n \quad \text{and} \quad 1 = \sum_{n \in B} |\alpha_n|.$$

We say that (g_n) is an l_1-normalized block-subsequence of (f_n) if each g_n is an l_1-normalized linear combination

$$g_n = \sum_{r \in B_n} \alpha_r f_r$$

and $\max B_n < \min B_{n+1}$.

LEMMA 1 (HAGLER-JOHNSON). *Let X be a Banach space and suppose that in X^* there is an infinite-dimensional subspace Y such that every weak* convergent sequence in Y is also norm convergent. Then there exists a bounded sequence (f_n) in X^* such that*

$$\sup_{x \in \operatorname{ball} X} \limsup_{n \to \infty} \langle g_n, x \rangle = 1$$

for every l_1-normalized block subsequence (g_n) of (f_n).

The idea for Lemma 2 can also be found in [**3**].

LEMMA 2. *Suppose that (f_n) is as in Lemma 1. Let $p \geq 1$ be a natural number and let g_{ij} $(1 \leq i \leq p,\ j \in \omega)$ be l_1-normalized linear combinations of the f_n,*

$$g_{ij} = \sum_{r \in B_{ij}} \alpha_r f_r,$$

such that

$$B_{ij} \cap B_{kl} = \varnothing \quad \textit{unless } (i,j) = (k,l).$$

Then, for all $\eta < 1$, there exists an infinite subset M of ω, together with elements x^F $(F \subseteq \{1, 2, \ldots, p\})$ of $\operatorname{ball} X$, such that, for all $j \in M$.

$$\langle g_{i,j}, x^F \rangle \begin{cases} > \eta & \textit{if } i \in F, \\ < -\eta & \textit{if } i \notin F. \end{cases}$$

PROOF. For each $E \subseteq \{1, 2, \ldots, p\}$, we consider

$$h_j^E = \frac{1}{p} \left[\sum_{i \in E} g_{i,j} - \sum_{i \notin E} g_{i,j} \right].$$

By our assumptions about the $g_{i,j}$ some subsequence of (h_j^E) is an l_1-normalized block subsequence of (f_n). Thus, given E and $\varepsilon > 0$, we may find an infinite subset K of ω and $x^E \in \operatorname{ball} X$ such that

$$\langle h_j^E, x^E \rangle > 1 - \varepsilon$$

for all $j \in K$. By repeating this process 2^p times, we obtain an infinite subset L of ω and elements $x^F \in \operatorname{ball} X$ such that

$$\langle h_j^F, x^F \rangle > 1 - \varepsilon$$

for all F and all $j \in L$.

Now for any F, the set of j such that there exists $i \leq p$ with

$$|\langle g_{i,j}, x^F \rangle| > 1 + \varepsilon$$

is finite. Thus, we may delete a finite number of elements of L to obtain a set M such that

$$\langle h_j^F, x^F \rangle > 1 - \varepsilon \quad \text{and} \quad |\langle g_{ij}, x^F \rangle| \leq 1 + \varepsilon$$

for all $j \in M$ and all $F \subseteq \{1, 2, \ldots, p\}$. It is easy to see that these inequalities imply

$$\langle g_{i,j}, x^F \rangle > 1 - (2p - 1)\varepsilon$$

when $i \in F$, and

$$\langle g_{i,j}, x^F \rangle < -1 + (2p - 1)\varepsilon$$

when $i \notin F$. So, given an appropriate initial choice of ε, we have the result.

As a first step towards proving the theorem, let us see what we need to do in order to define an isometry from $L_1(\{0,1\}^I)$ into a Banach space Y. We write $\mathcal{P}_f(I)$ for the set of all finite subsets of I, $\mathcal{F}_I$ for the union $\bigcup \{0,1\}^A \colon A \in \mathcal{P}_f(I)\}$ and, for $\alpha \in \mathcal{F}_I$, we let e_α be the element of $L_1(\{0,1\}^I)$ defined by

$$e_\alpha(z) = \begin{cases} 2^{\# \operatorname{dom} \alpha} & \text{if } z \mid \operatorname{dom} \alpha = \alpha, \\ 0 & \text{if not.} \end{cases}$$

The linear span of the e_α ($\alpha \in \mathcal{F}_I$) is dense in $L_1(\{0,1\}^I)$ and we have

$$e_\alpha = s^{\#A - \#B} \sum_{\substack{\beta \in \{0,1\}^B \\ \beta | A = \alpha}} e_\beta$$

when $A = \operatorname{dom} \alpha \subset B$. The following lemma is easy to prove.

LEMMA 3. *Let Y be a Banach space, I be a set, and let $(h_\alpha)_{\alpha \in \mathcal{F}_I}$ be a family of elements of Y. Then there exists a bounded linear operator $T \colon L_1(\{0,1\}^I) \to Y$ with $Te_\alpha = h_\alpha$ ($\alpha \in \mathcal{F}_I$) if and only if the family (h_α) is bounded and the following condition is satisfied:*

(a)
$$h_\alpha = 2^{\#A - \#B} \sum_{\substack{\beta \in \{0,1\}^B \\ \beta | A = \alpha}} h_\beta$$

whenever $\operatorname{dom} \alpha = A \subset B \in \mathcal{P}_f(I)$.

The operator T is isometric if and only if we have

(b)
$$\left\| \sum_{\alpha \in \{0,1\}^A} \theta(\alpha) h_\alpha \right\| = \sum_{\alpha \in \{0,1\}^A} |\theta(\alpha)|$$

whenever $A \in \mathcal{P}_f(I)$ and $\theta \colon \{0,1\}^A \to \mathbf{R}$.

In the proof of the theorem, we shall obtain vectors $h(\alpha)$ as weak* limits of certain $h_n(\alpha)$. The key combinatorial idea is that of a dyadic system.

Let D_j ($j \in \omega$) be the sequence of sets $D_0 = \{1\}$, $D_1 = \{2,3\}, \ldots, D_j = \{2^j, 2^j+1, \ldots, 2^{j+1}-1\}, \ldots$ and let I be a set. Suppose that, for each $a \in I$, $P_j(a)$ is a subset of D_j. For each finite subset A of I, let $M(A)$ be the set of all $j \in \omega$ such that, for all $\alpha \in \{0,1\}^A$, the intersection

$$\bigcap_{\substack{a\in A\\ \alpha(a)=0}} P_j(a) \cap \bigcap_{\substack{a\in A\\ \alpha(a)=1}} (D_j \setminus P_j(a))$$

has exactly $2^{j-\#A}$ elements. We say that $(P_j(a))$ forms a *dyadic system* over I if $\omega \setminus M(A)$ is finite for each finite $A \subseteq I$.

Let us note that there exists a dyadic system over a set I with $\#I = \mathfrak{c}$.

LEMMA 4. *There exists a dyadic system $(P_j(z))$ indexed by $z \in \{0,1\}^\omega$.*

PROOF. First of all, fix, for each $j > 0$, a family $Q_{j,r}$ ($0 \le r < j$) of subsets of D_j such that

$$\# \bigcap_{r\in H} Q_{j,r} = 2^{j-\#H}$$

for every $H \subseteq \{0, 1, \ldots, j-1\}$. The obvious choice is

$$Q_{j,r} = \{2^j + m \colon 0 \le m < 2^j \ \&\ [2^{-r}m] \text{ is even}\}.$$

We shall put

$$P_j(z) = Q_{j,r(z,j)}$$

for suitably chosen $r(z,j)$. What we need is that, for distinct $w, z \in \{0,1\}^\omega$, $r(z,j) = r(w,j)$ for only finitely many values of j. This can be achieved if, for example, we put $l = [\log_2 j]$ and

$$r(z,j) = \sum_{m=0}^{l-1} z_m 2^m.$$

PROOF OF THE THEOREM. Let $(P_j(\xi))_{\xi\in\mathfrak{p}}$ be a dyadic system over $\mathfrak{p}$, where we regard $\mathfrak{p}$ as the set of ordinals $\xi < \mathfrak{p}$. Such a system exists since $\mathfrak{p} \le \mathfrak{c}$. For $A \in \mathcal{P}_f(\mathfrak{p})$, $\alpha \in \{0,1\}^A$ write

$$D_j(\alpha) = \bigcap_{\substack{a\in A\\ \alpha(a)=0}} P_j(a) \cap \bigcap_{\substack{a\in A\\ \alpha(a)=1}} (D_j \setminus P_j(a))$$

and

$$h_j(\alpha) = \frac{1}{\#D_j(\alpha)} \sum_{n\in D_j(\alpha)} f_n.$$

We shall use transfinite induction to define infinite subsets M_ξ of ω ($\xi \in \mathfrak{p}$) and elements $x(F)$ of ball X ($F \subseteq \{0,1\}^A$, $A \in \mathcal{P}_f(\mathfrak{p})$). These will have the properties:

(i) $M_\eta \setminus M_\xi$ is finite when $\xi < \eta \in \mathfrak{p}$.

(ii) For all $\xi \in \mathfrak{p}$, all $A \in \mathcal{P}_f(\xi)$, and all $F \subseteq \{0,1\}^A$, there is a finite $L \subset M_\xi$ such that, for $j \in M_\xi \setminus L$,

$$\langle h_j(\alpha), x(F)\rangle \begin{cases} > 1 - 1/\#A & \text{if } \alpha \in F, \\ < -1 + 1/\#A & \text{if } \alpha \in \{0,1\}^A \setminus F. \end{cases}$$

To do the recursive construction, suppose that M_ξ and $x(F)$ have been defined already for all $\xi < \eta$ and all $F \subseteq \{0,1\}^A$ ($A \in \mathcal{P}_f(\xi)$). If η is a limit ordinal, we use the fact that $\eta < \mathfrak{p}$ to find an infinite $M_\eta \subset \omega$ with $M_\eta \setminus M_\xi$ finite for all $\xi < \eta$. Evidently, no new $x(F)$ need to be defined in this case.

Next, let η be a successor ordinal, $\eta = \varsigma + 1$. We have to find elements $x(F)$ ($F \subseteq \{0,1\}^{A\cup\{\varsigma\}}$, $A \in \mathcal{P}_f(\varsigma)$), and obtain the subsequence $M_{\varsigma+1}$. We note that, given $A \in \mathcal{P}_f(\varsigma)$ and an infinite subset M of ω, we can apply Lemma 2 (with $p = 2^{\#(A\cup\{\varsigma\})}$) to obtain an infinite subset M' of M and vectors $x(F) \in \operatorname{ball} X$ ($F \subseteq \{0,1\}^{A\cup\{\varsigma\}}$) such that, for $j \in M'$,

$$(*) \qquad \langle h_j(\alpha), x(F)\rangle \begin{cases} > 1 - 1/\#A & \text{if } \alpha \in F, \\ < -1 + 1/\#A & \text{if not.} \end{cases}$$

We now well-order $\mathcal{P}_f(\varsigma)$ and repeatedly use the above observation, together with the fact that $\#\mathcal{P}_f(\varsigma) < \mathfrak{p}$. We obtain in this way an infinite subset $M_{\varsigma+1}$ of M_ς, together with elements $x(F)$ of $\operatorname{ball} X$ and finite subsets $L(F)$ of $M_{\varsigma+1}$ (for $F \subseteq \{0,1\}^{A\cup\{\varsigma\}}$, $A \in \mathcal{P}_f(\varsigma)$), in such a way that $(*)$ holds whenever $j \in M_{\varsigma+1} \setminus L(F)$.

Once the sets M_ξ and the vectors $x(F)$ have been obtained, we let $\mathcal{U}$ be an ultrafilter on ω which contains all the sets M_ξ and all the cofinite sets. We define $h(\alpha) = \mathrm{w}^* \lim_{j\to\mathcal{U}} h_j(\alpha)$ and only have to check that (a) and (b) of Lemma 3 are satisfied.

If $A \subset B \in \mathcal{P}_f(\mathfrak{p})$ and $\alpha \in \{0,1\}^A$ then for all j in the cofinite set $M(B)$, we have

$$h_j(\alpha) = 2^{\#A - \#B} \sum_{\substack{\beta\in\{0,1\}^B \\ \beta|A=\alpha}} h_j(\beta),$$

because of the definition of dyadic system.

Taking the limit as $j \to \mathcal{U}$ gives (2).

Now if $A \in \mathcal{P}_f(\mathfrak{p})$ and $\theta\colon \{0,1\}^A \to \mathbf{R}$ we may fix any $B \in \mathcal{P}_f(\mathfrak{p})$ with $B \supseteq A$ and consider

$$G = \{\beta \in \{0,1\}^B : \theta(\beta \mid A) > 0\}.$$

It is easy to check that, if $B \subseteq \xi \in \mathfrak{p}$,

$$\left\langle \sum_{\alpha\in\{0,1\}^A} \theta(\alpha) h_j(\alpha), x(G) \right\rangle \geq (1 - 1/\#B) \sum_\alpha |\theta(\alpha)|$$

for all but finitely many $j \in M_\xi$.

This tells us that

$$\left\| \sum \theta(\alpha) h(\alpha) \right\| \geq (1 - 1/\#B) \sum |\theta(\alpha)|.$$

Since $\#B$ may be chosen to be an arbitrarily large integer, we have proved (b).

REMARKS. Although the proof has been presented here for the cardinal $\mathfrak{p}$, the only point of interest of the result lies in the case $\mathfrak{p} = \omega_1$. If $\mathfrak{p} > \omega_1$, then the result of **[4]** is in two respects stronger than the one given here. First, the hypothesis in **[4]** is that there exists in X^* a bounded sequence with no weak* convergent *convex* block subsequence; here we need "l_1-normalized" instead of "convex." Secondly, the conclusion in **[4]**, that X contains $l_1(\mathfrak{p})$, implies that X^* has a subspace isometric to $C(\{0,1\}^{\mathfrak{p}})^*$. I do not know whether it is true unconditionally that the dual of a nonreflexive Grothendieck space contains $C(\{0,1\}^{\omega_1})^*$. I also do not know whether the convex block hypothesis of [4] suffices for an unconditional result.

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