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THE l_1-COMPLETION OF A METRIC COMBINATORIAL ∞-MANIFOLD

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ABSTRACT. Let K be a simplicial complex. The realization $|K|$ of K admits the metric

$$d_1(x,y) = \sum_{v \in K^0} |x(v) - y(v)|,$$

where $x(v)$ and $y(v)$, $v \in K^0$, are the barycentric coordinates of x and y respectively. The completion of the metric space $(|K|, d_1)$ is called the l_1-completion and is denoted by $\overline{|K|}^{l_1}$. In this paper, we prove that $\overline{|K|}^{l_1}$ is an l_2-manifold if and only if K is a combinatorial ∞-manifold.

0. Introduction. Let K be a simplicial complex. The realization $|K|$ of K admits the metric

$$d_1(x,y) = \sum_{v \in K^0} |x(v) - y(v)|,$$

where $x(v)$ and $y(v)$, $v \in K^0$, are the barycentric coordinates of x and y respectively. The topology induced by d_1 is the *metric topology* of $|K|$ and the space $|K|$ with this topology is denoted by $|K|_m$. The completion of the metric space $(|K|, d_4)$ is called the *l_1-completion* of $|K|_m$ and is denoted by $\overline{|K|}^{l_1}$. In [**Sa**$_1$], the author proved that $\overline{|K|}^{l_1}$ is an ANR and the inclusion $i\colon |K|_m \subset \overline{|K|}^{l_1}$ is a fine homotopy equivalence, that is, for each open cover $\mathcal{U}$ of $\overline{|K|}^{l_1}$ there is a map $f\colon \overline{|K|}^{l_1} \to |K|_m$ such that $i \circ f$ is $\mathcal{U}$-homotopic to id and $f \circ i$ is $i^{-1}(\mathcal{U})$-homotopic to id, and conjectured that $\overline{|K|}^{l_1}$ is an l_2-manifold if K is a combinatorial ∞-manifold. Here a *combinatorial ∞-manifold* is a countable simplicial complex such that the star of each vertex is combinatorially equivalent to the countable-infinite full simplicial complex, namely a ∞-simplex Δ^∞ (see [**Sa**$_1$]). An *l_2-manifold* is a separable manifold modeled on the Hilbert space l_2. Let l_2^f be the linear span of the natural orthonormal basis of l_2. A separable manifold modeled on the space l_2^f is called an *l_2^f-manifold.* In [**Sa**$_2$, **Sa**$_3$], it was shown that K is a combinatorial ∞-manifold if and only if $|K|_m$ is an l_2^f-manifold. The main result of this paper is the following

MAIN THEOREM. *A simplicial complex K is a combinatorial ∞-manifold if and only if $\overline{|K|}^{l_1}$ is an l_2-manifold.*

1. Preliminaries. Let X be a metric space with a metric d. A closed subset A of X is called a *Z-set in X* if for each $\varepsilon > 0$ and each map $f\colon I^n \to X$, $n \in \mathbf{N}$,

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there is a map $g\colon I^n \to X\backslash A$ with $d(f,g) < \varepsilon$. An *(f.d.) cap set for X* is a subset M of X such that $M = \bigcup_{n=1}^{\infty} M_n$ where $M_1 \subset M_2 \subset \cdots$ is a tower of (finite-dimensional) compact Z-sets in X such that for each $\varepsilon > 0$, each $m \in \mathbf{N}$, and each (finite-dimensional) compact subset A of X, there is an $n \in \mathbf{N}$ and an embedding $h\colon A \to M_n$ such that $h|A \cap M_m = \mathrm{id}$ and $d(h, \mathrm{id}) < \varepsilon$. Let s denote the pseudo-interior of the Hilbert cube $Q = I^\omega$, that is, $s = \overset{\circ}{I}{}^\omega$, where $I = [0,1]$ and $\overset{\circ}{I} = (0,1)$. The pseudo-boundary $Q\backslash s$ is a cap set for Q. (Refer to [**An**] and [**Ch$_1$**].) A *Q-manifold* is a separable manifold modeled on the Hilbert cube Q. T. A. Chapman [**Ch$_1$**] proved that if M is a cap set for a Q-manifold X, then $X\backslash M$ is an l_2-manifold.

1.1 LEMMA. *Let X be a locally compact ANR with a metric d and let $M = \bigcup_{n=1}^{\infty} M_n \subset X$ such that each M_n is a Q-manifold and a Z-set in M_{n+1}, and*

(∗) *for each $\varepsilon > 0$, each $m \in \mathbf{N}$, and each compact subset A of X, there is an $n \in \mathbf{N}$ and a map $f\colon A \to M_n$ such that $f(A \cap M_m) \subset M_m$ and $d(f, \mathrm{id}) < \varepsilon$.*

Then X is a Q-manifold and M is a cap set for X, hence $X\backslash M$ is an l_2-manifold.

PROOF. It is easy to see that X is a Q-manifold by using Toruńczyk's characterization of Q-manifolds [**To**]. Since each compact set in a Q-manifold has a compact Q-manifold neighborhood (this can be easily seen by using [**Ch$_2$**, Theorems 37.2 and 28.1]), we can write $M_n = \bigcup_{i=1}^{\infty} M_{n,i}$, $n \in \mathbf{N}$, where each $M_{n,i}$ is a compact Q-manifold, and

$$M_{n,i} \subset \operatorname{int}_{M_n} M_{n,i+1} \cap \operatorname{int}_{M_{n+1}} M_{n+1,i}.$$

Then observe that $M = \bigcup_{n=1}^{\infty} M_{n,n}$. Since M_n is a Z-set in M_m for any $m > n$, it follows from (∗) that $M_{n,n}$ is a Z-set in X. Let A be a compact subset of X and let $\varepsilon > 0$ and $m \in \mathbf{N}$. Since M_m is an ANR, there is a $\delta > 0$ such that any map $f\colon A \cap M_{m,m} \to M_m$ with $d(f, \mathrm{id}) < \delta$ is $\varepsilon/3$-homotopic to the inclusion $A \cap M_{m,m} \subset M_m$. By (∗), we have a map $f\colon A \to M_{n'}$ for some $n' > m$ such that $f(A \cap M_m) \subset M_m$ and $d(f, \mathrm{id}) < \delta$. Then $f|A \cap M_{m,m}$ is $\varepsilon/3$-homotopic to the inclusion $A \cap M_{m,m} \subset M_m$. Since $M_{n'}$ is an ANR, we have a map $g\colon A \to M_{n'}$ such that $g|A \cap M_{m,m} = \mathrm{id}$ and $d(g,f) < \varepsilon/3$ by the Homotopy Extension Theorem [**Hu**, Chapter IV, Theorem 2.2 and its proof]. From compactness of $g(A)$, $g(A) \subset M_{n',n}$ for some $n \geq n'$ because

$$M_{n'} = \bigcup_{i=1}^{\infty} \operatorname{int}_{M_{n'}} M_{n',i}.$$

Then we have $g(A) \subset M_{n,n}$. Since M_m is a Z-set in M_n and

$$A \cap M_{m,m} \subset M_m \cap \operatorname{int}_{M_n} M_{n,n},$$

$A \cap M_{m,m}$ is a Z-set in $M_{n,n}$. By the Z-Embedding Approximation Theorem [**Ch$_2$**, Lemma 18.1], we have an embedding $h\colon A \to M_{n,n}$ such that $h|A \cap M_{m,m} = g|A \cap M_{m,m} = \mathrm{id}$ and $d(h,g) > \varepsilon/3$, hence $d(h, \mathrm{id}) < \varepsilon$. Therefore $M = \bigcup_{n=1}^{\infty} M_{n,n}$ is a cap set for X. □

2. Proof of the Main Theorem. By S_1^+, we denote the set of all points in the unit sphere of the Banach space l_1 having nonnegative coordinates, that is,

$$S_1^+ = \left\{ (x_i)_{i\in\mathbf{N}} \;\middle|\; \sum_{i=1}^{\infty} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \in \mathbf{N} \right\}.$$

Clearly $S_1^+ \subset Q$ as sets. It is well known and easy to see that the topology on S_1^+ induced by the norm $\|\cdot\|_1$ of l_1 coincides with the product topology. Thus $S_1^+ \subset Q$ as topological spaces.

Let K be a countable simplicial complex with vertices v_i, $i \in \mathbf{N}$. Identifying $x \in |K|$ with $(x_i)_{i\in\mathbf{N}} = (x(v_i))_{i\in\mathbf{N}} \in S_1^+$, we consider $|K|_m \subset S_1^+ \subset Q$. The l_1-completion $\overline{|K|}^{l_1}$ of $|K|_m$ is $\mathrm{cl}_{S_1^+}|K|$ because d_1 is the metric induced by the norm $\|\cdot\|_1$ and S_1^+ is closed in l_1. On the other hand $|K|_m$ has the local-compactification

$$\overline{|K|}^{Q^*} = \mathrm{cl}_{Q\setminus\{0\}}|K| = (\mathrm{cl}_Q|K|)\setminus\{0\},$$

where 0 denotes the point $(0, 0, \dots) \in Q$. Then $\overline{|K|}^{l_1}$ is a subspace of $\overline{|K|}^{Q^*}$. Moreover, by [**Sa**$_4$, Lemma 1.1; **Sa**$_5$, Lemma 1.1], we have the following

2.1 LEMMA. *For a countable simplicial complex K with no principal (maximal) simplex,*

$$\overline{|K|}^{Q^*} = (0,1] \cdot \overline{|K|}^{l_1} = \{tx | x \in \overline{|K|}^{l_1},\ t \in (0,1]\}.$$

In [**Sa**$_5$], the author proved that for a combinatorial ∞-manifold K, $\overline{|K|}^{Q^*}$ is a $[0,1)$-stable Q-manifold containing $|K|_m$ as an f.d. cap set. Here we say that a Q-manifold X is $[0,1)$-*stable* if $X \times [0,1)$ is homeomorphic to X. Now we prove moreover that $\overline{|K|}^{Q^*} \setminus \overline{|K|}^{l_1}$ is a cap set for $\overline{|K|}^{Q^*}$.

2.2 THEOREM. *If K is a combinatorial ∞-manifold, then $\overline{|K|}^{Q^*}$ is a $[0,1)$-stable Q-manifold, $|K|_m$ is an f.d. cap set for $\overline{|K|}^{Q^*}$, and $\overline{|K|}^{Q^*} \setminus \overline{|K|}^{l_1}$ is a cap set for $\overline{|K|}^{Q^*}$.*

PROOF. As mentioned above, the first two properties have been proved in [**Sa**$_5$]. To see the last property, we use Lemma 1.1. From 2.1, it follows that

$$\overline{|K|}^{Q^*} \setminus \overline{|K|}^{l_1} = \bigcup_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) \cdot \overline{|K|}^{Q^*}.$$

Each $(1 - 1/n) \cdot \overline{|K|}^{Q^*}$ is a Q-manifold because it is homeomorphic to $\overline{|K|}^{Q^*}$. To see that $(1 - 1/n) \cdot \overline{|K|}^{Q^*}$ is a Z-set in $(1 - 1/(n+1)) \cdot \overline{|K|}^{Q^*}$, it suffices to see that $t \cdot \overline{|K|}^{Q^*}$ is a Z-set in $\overline{|K|}^{Q^*}$ for $0 < t < 1$. In the proof of the Main Theorem of [**Sa**$_5$], it is proved that for each compact subset A of $\overline{|K|}^{Q^*}$ and each $\varepsilon > 0$, there is a map $f \colon A \to |K|_m$ with $d_Q(f, \mathrm{id}) < \varepsilon$, where d_Q is the metric for Q defined by

$$d_Q(x,y) = \sup_{i\in\mathbf{N}} \min\{|x_i - y_i|, i^{-1}\}.$$

For $0 < t < 1$, $|K| \cap t \cdot \overline{|K|}^{Q^*} = \emptyset$. Then it follows that $t \cdot \overline{|K|}^{Q^*}$ is a Z-set in $\overline{|K|}^{Q^*}$. Condition $(*)$ of Lemma 1.1 is clearly satisfied. Then the result follows from Lemma 1.1. □

PROOF OF THE MAIN THEOREM. From the above theorem, it follows that if K is a combinatorial ∞-manifold, then $\overline{|K|}^{l_1}$ is an l_2-manifold. Conversely assume that $\overline{|K|}^{l_1}$ is an l_2-manifold. Let A be a simplex of K. Since any compact subset of an l_2-manifold is a Z-set, A is a Z-set in $\overline{|K|}^{l_1}$. By [**Sa**$_4$, Theorem 0.1], the inclusion $|K|_m \subset \overline{|K|}^{l_1}$ is a fine homotopy equivalence. Then it follows that A is a Z-set in $|K|_m$. By the result of [**Sa**$_3$], K is a combinatorial ∞-manifold. This completes the proof. □

Combining the "only if" part of the main theorem with [**Sa**$_4$, Corollary 0.9], we have

2.3 COROLLARY. *For a combinatorial ∞-manifold K, $\overline{|K|}^{l_1}$ is an l_2-manifold which contains $|K|_m$ as an f.d. cap set.*

Thus Conjecture 0.8 in [**Sa**$_4$] is true. By Henderson [**He**], l_2-manifolds are topologically classified by homotopy type. Then from the topological uniqueness of f.d. cap sets for an l_2-manifold [**Ch**$_1$], we have

2.4 COROLLARY. *If two combinatorial ∞-manifold K and L have the same homotopy type, then the pairs $(\overline{|K|}^{l_1}, |K|_m)$ and $(\overline{|L|}^{l_1}, |L|_m)$ are homeomorphic.*

3. Remarks. For a contractible combinatorial ∞-manifold K, $\mathrm{cl}_Q|K|$ is homeomorphic to Q by [**Sa**$_5$, Corollary 2.3]. As seen in the proof of Theorem 2.2, $t \cdot \overline{|K|}^{Q^*} = (t \cdot \mathrm{cl}_Q|K|)\backslash\{0\}$ is a Z-set in $\overline{|K|}^{Q^*} = (\mathrm{cl}_Q|K|)\backslash\{0\}$ for $0 < t < 1$. Since $\{0\}$ is a Z-set in $\mathrm{cl}_Q|K|$, it follows that $t \cdot \mathrm{cl}_Q|K|$ is a Z-set in $\mathrm{cl}_Q|K|$ for $0 < t < 1$. Then as before, we can prove that

$$\mathrm{cl}_Q|K|\backslash \mathrm{cl}_{S_1^+}|K| = (\mathrm{cl}_Q|K|)\backslash\overline{|K|}^{l_1}$$

is a cap set for $\mathrm{cl}_Q|K|$. Thus we have

3.1 PROPOSITION. *For a contractible combinatorial ∞-manifold K, the pair $(\mathrm{cl}_Q|K|, \mathrm{cl}_{S_1^+}|K|) = (\mathrm{cl}_Q|K|, \overline{|K|}^{l_1})$ is homeomorphic to (Q, s).*

In connection with our results, the following general problem is very interesting.

3.2 PROBLEM. *Let M be an l_2^f-manifold with a metric d. Under what condition is the completion of the metric space (M, d) an l_2-manifold?*

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