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ON PREIMAGE KNOTS IN S^3

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ABSTRACT. In [**1**], Gordon proved that a certain sequence of preimage knots in S^3 is finite. In this paper, we prove that every sequence of preimage knots is finite by using the Gromov invariant for knots.

Introduction. Let V be a solid torus and l an essential loop in $\operatorname{int} V$. Here l is *essential* in V if ∂V is incompressible in $V - l$ and if l is not isotopic in V to a core c of V. Let $f: V \to S^3$ (resp. $g: V \to S^3$) be an embedding such that $f(c)$ (resp. $g(c)$) is knotted (resp. unknotted) in S^3, and let $T = f(\partial V)$. We say $f(c)$ is the *companion* of $f(l)$ for T, $f(l)$ is the *satellite* of $f(c)$ for T, and $g(l)$ is a *preimage* of $f(l)$ for T. Here we do not fix the twistings of embeddings f,g in any manner, so a preimage of $f(l)$ for T is not determined uniquely from $f(l)$ and T. For two knots K_0, K in S^3, an "inequality" $K_0 < K$ will mean that K_0 is a preimage of K (for some torus in S^3). We say two knots K_1, K_2 have the *same knot type* if there exists a homeomorphism $h: S^3 \to S^3$ such that $h(K_1) = K_2$. We denote by $K_1 \cong K_2$ that K_1 and K_2 have the same knot type.

The following is a fundamental result on preimage knots.

THEOREM 1. *Let K_0, K be knots in S^3. If $K_0 < K$, then $K_0 \not\cong K$.*

To prove Theorem 1, we use the Gromov invariant for knots, that is, the Gromov invariant of knot exteriors. In [**9**], Thurston proved that the set of values of the Gromov invariant on Haken manifolds with toral boundaries is a closed well-ordered set of $\mathbf{R}_+$. Using this fact, we shall prove the following theorem (cf. Gordon [**1**, §5]).

THEOREM 2. *Every sequence of preimage knots in S^3 is finite, that is, there exists no infinite sequence $\{K_n\}_{n=1}^{\infty}$ of knots such that $K_1 > K_2 > \cdots > K_n > K_{n+1} > \cdots$.*

Let V be a solid torus with a core c and l an essential loop in V. Let $f_1, f_2: V \to S^3$ be two embeddings. In [**6**], Kouno proved that, if $f_1(l) \cong f_2(l)$ and if both $f_1(c)$ and $f_2(c)$ are knotted, then $f_1(c) \cong f_2(c)$. On the other hand, Theorem 1 above shows that, if $f_1(c)$ is unknotted and $f_2(c)$ is knotted, then $f_1(l) \not\cong f_2(l)$. Therefore, by Kouno's theorem [**6**] together with our theorem, the following corollary is obtained immediately.

COROLLARY. *With the notation as above if $f_1(l) \cong f_2(l)$, then $f_1(c) \cong f_2(c)$.*

1. Preliminaries. For fundamental notations on 3-manifolds, we refer to Jaco [**3**].

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Let M be a Haken manifold with toral boundary. According to Thurston's Uniformization Theorem (see Morgan [**7**]), there exists a set of mutually disjoint, incompressible tori $T_1, \dots T_n$ (possibly empty) in M satisfying the following (*).

(*) Every T_i is not boundary parallel and, for each pair T_i, T_j, $i \neq j$, T_i is not parallel to T_j in M. Furthermore, for the closure P of each component of $M - T_1 \cup \cdots \cup T_n$, either P is Seifert-fibered or $\operatorname{int} P$ admits a complete hyperbolic structure of finite volume. The former case P is called a *Seifert piece* and the latter a *hyperbolic piece.*

We say $(M, T_1, \dots, T_n)$ is a *torus decomposition* of M if $\{T_i\}_{i=1}^n$ satisfies (*). A torus decomposition of M is *minimal* if it is one with the minimal number of tori among all torus decompositions of M. A minimal torus decomposition of every Haken manifold M with toral boundary is unique up to ambient isotopy (see Jaco-Shalen [**4**] and Johanson [**5**]). We denote by $\tau(M)$ the number of pieces in a minimal torus decomposition of M.

Let X be a topological space, and let $c = \sum_i r_i \sigma_i$ be a finite combination of singular k-simplices $\sigma_i \colon \Delta^k \to X$ with real coefficients r_i. We define the *norm* $\|c\|$ of c by $\sum_i |r_i| \geq 0$.

Let M be a compact, orientable 3-manifold with toral boundary. The *Gromov invariant* $\|M\|$ of M is given by

$$\inf\{\|z\|;\ z \text{ is a singular cycle representing } [M, \partial M]\},$$

where $[M, \partial M] \in H_3(M, \partial M, \mathbf{R})$ is the fundamental class of $(M, \partial M)$.

Though this definition of the Gromov invariant is given in a way different from that in [**9**, §6.5], Gromov proved that both definitions give the same invariant (see [**2**,§4.1]).

Let X be a subpolyhedron of a simplicial polyhedron Y. We denote by $N(X, Y)$ a regular neighborhood of X in Y.

For a knot K in S^3, $E(K)$ denotes the exterior of K, i. e., $E(K) = S^3 - \operatorname{int} N(K, S^3)$. As in [**8**], we define the Gromov invariant $\|K\|$ of a knot K in S^3 by $\|E(K)\|$.

Let S be a Seifert fibered space with nonempty, incompressible boundary and not homeomorphic to $T^2 \times I$. If $S \subset S^3$, then S cannot be a twisted I-bundle over the Klein bottle. Therefore, by [**3**, VI. 18], S has the unique Seifert fibered structure up to ambient isotopy. Note that the euler number $\chi_b(S)$ of the base 2-orbifold (see [**9**, Chapter 13]) of a fibration on S is negative.

Let K be a knot in S^3, and let $S_1, \dots, S_n$ be the Seifert pieces in a minimal torus decomposition of $E(K)$. We set $\chi_b(K) = \sum_{i=1}^n \chi_b(S_i)$. We define the complexity $c(K)$ of K to be the lexicographically ordered triple $(\|K\|, \tau(K), -\chi_b(K))$, where $\tau(K) = \tau(E(K))$.

2. Proofs of Theorems 1 and 2. Let V be a solid torus with a core c and l an essential loop in V. Let $f \colon V \to S^3$ (resp. $g \colon V \to S^3$) be an embedding such that $f(c)$ (resp. $g(c)$) is knotted (resp. unknotted) in S^3. We set $C = V - \operatorname{int} N(l, V)$, $K_0 = g(l)$, and $K = f(l)$. Hence $K_0 < K$. The exterior $E(K_0)$ (resp. $E(K)$) is homeomorphic to $X \cup_{g|\partial V} C$ (resp. $Y \cup_{f|\partial V} C$), where $X = S^3 - \operatorname{int} g(V)$ is a solid torus and $Y = S^3 - \operatorname{int} f(V)$ is the exterior of a knot $f(c)$.

First we prove the following lemma.

LEMMA. $c(K_0) < c(K)$.

To prove this lemma, we consider the torus decomposition of C. Let $\{T_i\}_{i=1}^n$ be a set of tori in C which defines a minimal torus decomposition of C. Let P be the closure of the component of $C - T_1 \cup \cdots \cup T_n$ such that $\partial P \supset \partial V$. Let T be the component of $\partial P - \partial V$ such that either $T = \partial C - \partial V$ or T separates the two tori $\partial C - \partial V$ and ∂V.

SUBLEMMA A. *If T is incompressible in $X \cup_{g|\partial V} P$, then $c(K_0) < c(K)$.*

PROOF. By the results in [**2**, **9**, and **8**], we have the following inequalities:

$$\|K_0\| = \|E(K_0)\| \leq \|E(K_0) - \operatorname{int}(X \cup_{g|\partial V} P)\| + \|X \cup_{g|\partial V} P\|, \tag{2.1}$$

$$\begin{aligned} \|K\| = \|E(K)\| &= \|E(K) - \operatorname{int}(Y \cup_{f|\partial V} P)\| + \|Y\| + \|P\| \\ &\geq \|E(K) - \operatorname{int}(Y \cup_{f|\partial V} P)\| + \|P\|. \end{aligned} \tag{2.2}$$

Since $E(K_0) - \operatorname{int}(X \cup_{g|\partial V} P)$ is homeomorphic to $E(K) - \operatorname{int}(Y \cup_{f|\partial V} P)$,

$$\|E(K_0) - \operatorname{int}(X \cup_{g|\partial V} P)\| = \|E(K) - \operatorname{int}(Y \cup_{f|\partial V} P)\|. \tag{2.3}$$

Since, by [**9**, Proposition 6.5.2], $\|X \cup_{g|\partial V} P\| \leq \|X\| + \|P\| = \|P\|$, the inequalities (2.1), (2.2) and the equality (2.3) imply $\|K_0\| \leq \|K\|$.

Now we need to consider the following two cases.

Case 1. P is a Seifert piece.

First we prove that $X \cup_{g|\partial V} P$ is Seifert-fibered. If not, any fiber l in ∂V of a fibration on P bounds a disk D in a solid torus X. There exists a (saturated)annulus A in P such that $\partial A \supset l$ and $l_0 = \partial A - l$ is a fiber in T. Hence l_0 bounds a disk $A \cup D$ in $X \cup_{g|\partial V} P$. Since l_0 is essential in T, T is compressible in $X \cup_{g|\partial V} P$, a contradiction. Thus $X \cup_{g|\partial V} P$ is Seifert-fibered and hence every component of $\partial(X \cup_{g|\partial V} P)$ is incompressible in $X \cup_{g|\partial V} P$ and so it is in $X \cup_{g|\partial V} C$. Thus $\{T_i\}_{i=1}^n$ defines a minimal torus decomposition of $X \cup_{g|\partial V} C$. Let $\{S_i\}_{i=1}^m$ be a set of tori in Y which defines a minimal torus decomposition of Y. Let Q be the closure of the component of $Y - S_1 \cup \cdots \cup S_m$ containing ∂Y.

Subcase 1-(i). $Q \cup_{f|\partial V} P$ is Seifert-fibered.

In this case, Q is also Seifert-fibered but neither a solid torus nor $T^2 \times I$. Since $-\chi_b(Q \cup_{f|\partial V} P) > -\chi_b(X \cup_{g|\partial V} P)$, $-\chi_b(K) > -\chi_b(K_0)$. Since $\tau(K) = \tau(C) + \tau(Y) - 1 \geq \tau(C) = \tau(K_0)$, we have $c(K) > c(K_0)$.

Subcase 1-(ii). $Q \cup_{f|\partial V} P$ is not Seifert-fibered.

Since $\tau(K) = \tau(C) + \tau(Y) > \tau(C) = \tau(K_0)$, $c(K) > c(K_0)$.

Case 2. P is a hyperbolic piece.

We denote the volume of a complete hyperbolic structure on $\operatorname{int} P$, which is uniquely determined by Mostow's Rigidity Theorem, by vol $(\operatorname{int} P)$. For a positive integer n, let $p\colon M \to S^3$ be an n-fold branched covering branched over a core of X. Since X is unknotted in S^3, M is homeomorphic to S^3. But we will not use the fact later. We set $\tilde{P} = p^{-1}(g(P))$ and $\tilde{X} = p^{-1}(X)$. Then $\operatorname{int} \tilde{P}$ admits a complete hyperbolic structure of finite volume and $\tilde{X}$ is a solid torus. If n is sufficiently large, then, by [**9**, Theorem 5.9], $\operatorname{int}(\tilde{X} \cup \tilde{P})$ admits a complete hyperbolic structure and, by [**9**, Theorem 6.5.6],

$$\operatorname{vol}(\operatorname{int}(\tilde{X} \cup \tilde{P})) < \operatorname{vol}(\operatorname{int} \tilde{P}). \tag{2.4}$$

Since $p|\tilde{X} \cup \tilde{P}: \tilde{X} \cup \tilde{P} \to X \cup_{g|\partial V} P$ is a proper degree n map, by the definition of the Gromov invariant, we have $\frac{1}{n}\|\tilde{X} \cup \tilde{P}\| \geq \|X \cup_{g|\partial V} P\|$. Since $p\,|\,\tilde{P}: \tilde{P} \to P$ is an n-fold (unbranched) covering, $\frac{1}{n}\|\tilde{P}\| = \|P\|$. By [**9**, Theorem 6.5.4], $\|\tilde{P}\| = \mathrm{vol}(\mathrm{int}\,\tilde{P})/v_3$ and $\|\tilde{X} \cup \tilde{P}\| = \mathrm{vol}(\mathrm{int}(\tilde{X} \cup \tilde{P}))/v_3$, where v_3 is the volume of a regular ideal simplex in $\mathbf{H}^3$. Therefore, by (2.4), we have $\|\tilde{P}\| > \|\tilde{X} \cup \tilde{P}\|$ and hence

$$\|P\| = \frac{1}{n}\|\tilde{P}\| > \frac{1}{n}\|\tilde{X} \cup \tilde{P}\| \geq \|X \cup_{g|\partial V} P\|. \tag{2.5}$$

By (2.1)–(2.3) and (2.5), we have $\|K_0\| < \|K\|$ and $c(K_0) < c(K)$. This completes the proof of Sublemma A.

SUBLEMMA B. *If T is compressible in $X \cup_{g|\partial V} P$, then $c(K_0) < c(K)$.*

PROOF. If $T = \partial C - \partial V$, then K_0 is unknotted and so $c(K_0) < c(K)$. If not, T is contained in $\mathrm{int}\,C$. Let Z be the union of the closures of all components (possibly empty) of $C - P$ which do not contain $\partial C - \partial V$. Since T is compressible in $X_1 = X \cup_{g|\partial V} (P \cup Z)$, X_1 is a solid torus. Since $T \subset \mathrm{int}\,C$, $C_1 = \overline{(C - P \cup Z)}$ is not empty. The union $V_1 = C_1 \cup N(l, V)$ is a solid torus containing l such that $K = f_1(l), K_0 = g_1(l)$, and $\tau(C_1) < \tau(C)$, where $f_1 = f\,|\,V_1$ and $g_1 = g\,|\,V_1$. Let c_1 be a core of V_1. Since X_1 is a solid torus, $g_1(c_1)$ is unknotted in S^3. Therefore the proof is completed by induction on $\tau(C)$.

The lemma and Theorem 1 are immediate from Sublemmas A and B. By Thurston [**9**, Corollary 6.6.3], the set of values of $\|K\|$ for all knots K in S^3 is a well-ordered set of $\mathbf{R}_+$. Therefore the set of values of $c(K)$ is also well-ordered. By this fact, Theorem 2 is also immediate from the lemma.

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