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A DISTINGUISHING EXAMPLE IN k-SPACES

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ABSTRACT. Not all locally compact spaces are k-spaces (that is, in the coreflective hull of the compact Hausdorff spaces).

1. The largest known cartesian closed coreflective full subcategory of Top is the coreflective hull, say $\mathcal{K}_3$, of the category of spaces variously known as exponentiable, exponential, core-compact, or quasi-locally compact: the spaces whose topology is a continuous lattice. In $\mathcal{K}_3$ is $\mathcal{K}_2$, the coreflective hull of the locally compact spaces; and in $\mathcal{K}_2$ is the coreflective hull $\mathcal{K}_1$ of the compact Hausdorff spaces. It has been unknown whether $\mathcal{K}_1 = \mathcal{K}_3$. This note distinguishes them; in fact, $\mathcal{K}_1 \neq \mathcal{K}_2$. That answers Problem 5 of Herrlich [**1**].

Problem 6, whether $\mathcal{K}_2 = \mathcal{K}_3$, remains. Also, the example is not sober, and thereby hangs another problem. Note that $\mathcal{K}_2$ and $\mathcal{K}_3$ certainly contain the same sober spaces, since every sober exponentiable space is locally compact [**2**].

If you want a T_1 example, help yourself, for passage to the smallest containing T_1 topology preserves local compactness and nonmembership in $\mathcal{K}_1$. (The example below is locally compact; anyway, the existence of examples implies the existence of locally compact examples, since "coreflective hull" is a closure operation.)

2. What prevents compact Hausdorff spaces (and even compact normal spaces) from generating $\mathcal{K}_2$ is this:

If a compact normal space has subsets S_α indexed by a non-σ-compact initial segment of the ordinals, the union of any proper initial segment of S_α's is closed, and the union U of all S_α is open, then U is closed.

PROOF. Suppose U is not closed. Let x_1 be a point of some S_{α_1}. $I_1 = \bigcup[S_\alpha : \alpha \leq \alpha_1]$ is closed and disjoint from the closed complement R of U, so they have disjoint neighborhoods N_1, Q_1. $N_1 \neq U$, since U, being nonclosed, meets Q_1. Inductively, having $x_1, \ldots, x_k, x_j$ in S_{α_j}, x_{j+1} outside a neighborhood N_j of I_j, and $N_k \neq U$ a neighborhood of $I_1 \cup \cdots \cup I_k$, choose x_{k+1} in $U - N_k$. I_{k+1} and R have disjoint neighborhoods, and the induction runs. But $\bigcup I_j$ is a countable union, hence proper and closed; it is covered by the open sets N_j, but by no finite number of them, which is absurd.

Consider the space X consisting of the countable ordinals and ω_1, with closed sets the countable initial segments, $\{\omega_1\}$, unions of two of those, and X. Every subspace is compact; for if nonempty, it has a least element, a neighborhood of which contains the rest of the subspace except perhaps ω_1. So X is locally compact. However, for any continuous map f from a compact Hausdorff space to X, by the

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lemma above, $f^{-1}(\{\omega_1\})$ is open. Thus each such f factors through X^*, which is X with $\{\omega_1\}$ made open; and $X \in \mathcal{K}_2 - \mathcal{K}_1$.

3. Let Y be a topological space and Y_1 the same set with the smallest containing T_1 topology, a subbase for which is given by the open sets of Y and the complements of singletons. By Alexander's Lemma, Y_1 is compact if Y is. (It is a bit easier than Alexander's Lemma.) Suppose Y is locally compact, and consider a basic neighborhood $W = U - \{x_1, \ldots, x_n\}$ of $p \in Y_1$. For $i \leq n$, if $p \notin \{x_i\}^-$ in Y, $Y - \{x_i\}$ is a neighborhood of p; intersecting, we have $W = V - F$ where V is a y-neighborhood of p and F is a (finite) set of points x_i with $p \in \{x_i\}^-$ in Y. Then V contains a compact (Y-) neighborhood N of p. $N - F$ is also compact in Y, for any open sets of Y covering it cover p and F. By the previous remark, $N - F$ is compact in Y_1. Finally, suppose $Y_1 \in \mathcal{K}_1$. The $\mathcal{K}_1$-coreflection Y^* of Y has a topology contained in the topology of Y_1 (since the continuous identity function $Y_1 \to Y$ factors through Y^*), and $p \in \{x\}^-$ in Y is preserved in the coreflection because of the map $[0,1] \to Y$ taking 0 to p and the rest to x. Thus $Y^* \to Y$ is a homeomorphism, and $Y \in \mathcal{K}_1$.

References

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