PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 3, July 1987

SHORTER NOTES

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A SIMPLE PROOF OF AN EXTENSION OF THE FUGLEDE-PUTNAM THEOREM

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ABSTRACT. A simple proof is given of the

THEOREM. *If A and B^* are hyponormal, then $\|AX - XB\|_2 \geq \|A^*X - XB^*\|_2$ for every X in the Hilbert-Schmidt class.*

In this note an operator means a bounded linear operator on a separable infinite-dimensional Hilbert space.

The following theorem has brought attention since it appeared in 1978.

THEOREM (BERBERIAN). *If A and B^* are hyponormal, then*

$$AX = XB \quad \textit{implies} \quad A^*X = XB^*$$

for every X in the Hilbert-Schmidt class [**1**].

In 1981, Takayuki Furuta gave an extension to the above theorem.

THEOREM (FURUTA). *If A and B^* are hyponormal, then*

$$\|AX - XB\|_2 \geq \|A^*X - XB^*\|_2$$

for every X in the Hilbert-Schmidt class. The equality holds when A and B are both nomal, where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm [**2**].

We shall show in this note that the assertions of these two theorems are immediate consequences of elementary properties of the Hilbert-Schmidt class.

It is known that the Hilbert-Schmidt class forms a two-sided ideal and is itself a Hilbert space with the inner product

$$(X, Y) = \sum_i (Xe_i, Ye_i) = \mathrm{Tr}(Y^*X) = \mathrm{Tr}(XY^*),$$

where X and Y are operators in the ideal and $\{e_i\}$ is any orthonormal basis in the Hilbert space under consideration. The elementary properties which are needed for

Received by the editors December 8, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47B10, 47B15, 47B20; Secondary 47A30.

Key words and phrases. Hyponormal operator, Hilbert-Schmidt class.

our proof are the following two: Let X and Y belong to the ideal and let A be an arbitrary operator. Then

(i) $(AX, Y) = (X, A^*Y)$, $(XA, Y) = (X, YA^*)$ [**3**, II, Lemma 5].

(ii) If $A \geq 0$, then $(AX, X) \geq 0$ and $(XA, X) \geq 0$ (indeed, for example, $(AX, X) = (A^{1/2}A^{1/2}X, X) = (A^{1/2}X, A^{1/2}X) \geq 0$).

PROOF OF FURUTA'S THEOREM.

$$\begin{aligned}
\|A^*X - XB^*\|_2 &= (A^*X, A^*X) - (XB^*, A^*X) - (A^*X, XB^*) + (XB^*, XB^*) \\
&= (AA^*X, X) - (AX, XB) - (XB, AX) + (XB^*B, X) \\
&\leq (A^*AX, X) - (AX, XB) - (XB, AX) + (XBB^*, X) \\
&= (AX, AX) - (AX, XB) - (XB, AX) + (XB, XB) \\
&= \|AX - XB\|_2. \quad \square
\end{aligned}$$

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