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INVARIANCE UNDER OPERATION $\mathcal{A}$

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ABSTRACT. The invariance under operation $\mathcal{A}$ of the families of sets having the classical Baire property, of Lebesgue measurable sets, and of Marczewski sets is established in a unified manner.

Marczewski has formulated a general theorem which simultaneously implies the invariance under the set-theoretical operation $\mathcal{A}$ of the family of Lebesgue measurable sets and the family of sets having the classical Baire property (see [**18**, **19**] and, for related matters, [**5**–**8**, **12**–**15**, **18**, **22**]). In [**21**] Marczewski further established the invariance under operation $\mathcal{A}$ of a new family of sets, which we call Marczewski sets, but utilized a different method of proof. By a suitable modification of Marczewski's general argument we unify these three results here and establish the invariance under operation $\mathcal{A}$ for any category base.

For the relevant definitions and properties of category bases used below see [**11**]. For additional classifications of sets invariant under operation $\mathcal{A}$ see [**10**].

THEOREM. *The family of sets having the Baire property with respect to any category base is invariant under operation $\mathcal{A}$.*

PROOF. We denote by $\mathbf{Z}$ the set of all infinite sequences $\nu = \langle \nu_1, \nu_2, \dots \rangle$ of natural numbers and by $\mathbf{N}^k$ the set of all k-tuples $\langle \nu_1, \nu_2, \dots, \nu_k \rangle$ whose terms are elements of the set $\mathbf{N}$ of natural numbers.

Let

$$S = \bigcup_{\nu \in \mathbf{Z}} \bigcap_{k=1}^{\infty} S_{\nu_1 \cdots \nu_k}$$

be the nucleus of a determinant system of sets $S_{\nu_1 \cdots \nu_k}$ which have the Baire property. The family of sets which have the Baire property being closed under finite intersections, we may assume, without loss of generality, that for each sequence $\nu = \langle \nu_1, \nu_2, \dots \rangle \in \mathbf{Z}$ and each $k \in \mathbf{N}$ we have

$$S_{\nu_1 \cdots \nu_{k+1}} \subset S_{\nu_1 \cdots \nu_k}.$$

(Otherwise, setting

$$S'_{\nu_1 \cdots \nu_k} = \bigcap_{p=1}^{k} S_{\nu_1 \cdots \nu_p}$$

for all $\nu \in \mathbf{Z}$ and all $k \in \mathbf{N}$, we obtain a determinant system of sets $S'_{\nu_1 \cdots \nu_k}$ having the Baire property which satisfies this inclusion and whose nucleus is also S.)

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In order to show that S has the Baire property it suffices to show that if A is any region in which S is abundant everywhere, then $A-S$ is a meager set. Assume therefore that A is such a region.

Suppose $k \in \mathbf{N}$ and $\alpha = \langle \nu_1, \dots, \nu_k \rangle \in \mathbf{N}^k$. Define

$$T_\alpha = \bigcup_{\mu \in \mathbf{Z}} \bigcap_{j=1}^{\infty} S_{\nu_1 \cdots \nu_k \mu_1 \cdots \mu_j}.$$

We proceed to define a particular maximal family (possibly empty) $\mathcal{M}_\alpha$ of disjoint subregions of A such that T_α is abundant everywhere in each region in $\mathcal{M}_\alpha$.

Let $\mathcal{N}_\alpha$ consist of all those regions in which T_α is either meager or abundant everywhere. Then $(\bigcup \mathcal{N}_\alpha, \mathcal{N}_\alpha)$ is a category base. Applying Lemma 2 of [**11**], we define $\mathcal{M}_\alpha^*$ to be a subfamily of $\mathcal{N}_\alpha$, consisting of disjoint regions, having the property that for every region $N \in \mathcal{N}_\alpha$ there is a region $M \in \mathcal{M}_\alpha^*$ such that $N \cap M$ contains a region in $\mathcal{N}_\alpha$. Set

$$\mathcal{M}_\alpha = \{M \in \mathcal{M}_\alpha^* \colon T_\alpha \text{ is abundant everywhere in } M\}.$$

Now defining $R_\alpha = S_\alpha \cap (\bigcup \mathcal{M}_\alpha)$, we have $R_\alpha \subset S_\alpha$. Set $Q = A - \bigcup_{m=1}^{\infty} R_m$ and, for each $k \in \mathbf{N}$ and $\langle \nu_1, \dots, \nu_k \rangle \in \mathbf{N}^k$, set

$$Q_{\nu_1 \cdots \nu_k} = R_{\nu_1 \cdots \nu_k} - \bigcup_{m=1}^{\infty} R_{\nu_1 \cdots \nu_k m}.$$

We then have

$$\begin{aligned} A - S = A - \bigcup_{\nu \in \mathbf{Z}} \bigcap_{k=1}^{\infty} S_{\nu_1 \cdots \nu_k} &\subset A - \bigcup_{\nu \in \mathbf{Z}} \bigcap_{k=1}^{\infty} R_{\nu_1 \cdots \nu_k} \\ &\subset \left(A - \bigcup_{m=1}^{\infty} R_m\right) \cup \left[\bigcup_{\nu \in \mathbf{Z}} \bigcap_{k=1}^{\infty} \left(R_{\nu_1 \cdots \nu_k} - \bigcup_{m=1}^{\infty} R_{\nu_1 \cdots \nu_k m}\right)\right] \\ &= Q \cup \left(\bigcup_{k=1}^{\infty} \bigcup_{\alpha} Q_\alpha\right), \end{aligned}$$

where α varies over all sequences $\alpha = \langle \nu_1, \dots, \nu_k \rangle \in \mathbf{N}^k$ for each $k \in \mathbf{N}$. Now, the totality of sets Q_α is countable. Hence, in order to show that $A - S$ is a meager set, we have only to show Q and all the sets Q_α are meager sets.

Suppose Q is not meager. Being a subset of A, the set Q is abundant everywhere in a region $B \subset A$. From the inclusion $S \subset \bigcup_{n=1}^{\infty} T_n$ and the fact that S is abundant in B, it follows that there is an index n_1 such that T_{n_1} is abundant in B. There is then a subregion N of B in which T_{n_1} is abundant everywhere. According to the definition of the family $\mathcal{M}_{n_1}^*$, there exists a region $M \in \mathcal{M}_{n_1}^*$ such that $N \cap M$ contains a region C in which T_{n_1} is abundant everywhere. As $T_{n_1} \subset S_{n_1}$, the set S_{n_1} is also abundant everywhere in C. Now, we have

$$S_{n_1} \cap C \subset R_{n_1} \subset \bigcup_{m=1}^{\infty} R_m,$$

which implies

$$Q \subset X - \bigcup_{m=1}^{\infty} R_m \subset X - (S_{n_1} \cap C).$$

Hence, Q being abundant everywhere in C, the set $X - (S_{n_1} \cap C)$ is abundant everywhere in C. Because S_{n_1} is also abundant everywhere in C and both S_{n_1} and $X - (S_{n_1} \cap C)$ have the Baire property, the set

$$S_{n_1} \cap [X - (S_{n_1} \cap C)] = S_{n_1} - C$$

is abundant in C. But this is impossible! We conclude Q must be a meager set.

Suppose $k \in \mathbf{N}$ and $\alpha = \langle \nu_1, \dots, \nu_k \rangle \in \mathbf{N}^k$. To show that Q_α is a meager set, we assume to the contrary that Q_α is abundant. Then Q_α is abundant everywhere is some region D.

The set Q_α is abundant everywhere is some region $B \in \mathcal{N}_\alpha$. For, if T_α is meager in D then $D \in \mathcal{N}_\alpha$, so we may take $B = D$. Whereas, if T_α is abundant in D then T_α is abundant everywhere is some region $B \subset D$, so $B \in \mathcal{N}_\alpha$ and Q_α is abundant everywhere in B.

The set T_α must also be abundant everywhere in B. For, suppose T_α is meager in some region $B' \subset B$. Then there exists a region $M^* \in \mathcal{M}_\alpha^*$ and a region $B'' \in \mathcal{N}_\alpha$ such that $B'' \subset B' \cap M^*$. The set T_α cannot be abundant everywhere in M^* and, consequently, $M^* \notin \mathcal{M}_\alpha$. The regions in $\mathcal{M}_\alpha^*$ being disjoint, we have $M^* \cap (\bigcup \mathcal{M}_\alpha) = \varnothing$. Since $Q_\alpha \subset \bigcup \mathcal{M}_\alpha$, we have $B'' \cap Q_\alpha = \varnothing$. This implies that Q_α is not abundant everywhere in B, a contradiction!

Having thus established that T_α is abundant everywhere in B, we can replace Q with Q_α, S with T_α, T_n with $T_{\nu_1 \cdots \nu_k n}$, n_1 with $\langle \nu_1, \dots, \nu_k, n_1 \rangle$, and R_m with $R_{\nu_1 \cdots \nu_k m}$ in the above argument, to obtain the conclusion that Q_α must be a meager set.

COROLLARY (cf. [**9**]). *There is no category base consisting of sets of real numbers for which the sets with the Baire property coincide with the linear Borel sets.*

REMARK. Concerning more general operations which preserve the classical Baire property and measurability see [**1**–**4**, **16**, **17**].

REFERENCES

1. A. A. Ljapunov, *On set-theoretical operations which preserve measurability*, Dokl. Akad. Nauk SSSR (N.S.) **65** (1949), 609–612.
2. ——, *On δs-operations preserving measurability and the Baire property*, Mat. Sb. (N.S.) **24** (**66**) (1949), 119–127.
3. ——, *R-sets*, Trudy Mat. Inst. Steklov. **40** (1953), 1–68.
4. ——, *On the method of transfinite indices in the theory of operations on sets*, Trudy Mat. Inst. Steklov. **133** (1973), 132–148; English transl., Proc. Steklov Inst. Math. **133** (1973), 133–149 (esp. §5).
5. N. Lusin, *Sur la classification de M. Baire*, C. R. Acad. Sci. Paris **164** (1917), 91–94.
6. ——, *Sur les ensembles analytiques*, Fund. Math. **10** (1927), 1–95 (esp. pp. 25–28).
7. N. Lusin and W. Sierpiński, *Sur quelques propriétés des ensembles* (*A*), Bull. Intern. Acad. Sci. Cracovie (1918), 35–48.
8. ——, *Sur un ensemble non mesurable B*, J. Math. (9) **2** (1923), 53–72.
9. J. C . Morgan II, *On the Borel σ-field and the Baire property*, Proc. Amer. Math. Soc. **90** (1984), 250–252.
10. ——, *On the general theory of point sets*, Real Anal. Exchange **9** (1984), 345–353.
11. ——, *Measurability and the abstract Baire property*, Rend. Circ. Mat. Palermo (2) **34** (1985), 234–244.
12. O. Nikodým, *Sur une propriété de l'opération A*, Fund. Math. **7** (1925), 149–154.

13. ——, *Sur quelques propriétés de l'opération A*, Spraw. Towarz. Nauk. Warszaw. Wyd. III **19** (1926), 294–298.
14. C. A. Rogers, *Hausdorff measures*, Cambridge Univ. Press, Cambridge, 1970 (esp. Chapter 1, §7).
15. S. Saks, *Theory of the integral*, Monografie Mat., Tom VII (2nd ed.), Hafner, New York, 1937 (esp. Chapter II, §5).
16. K. Schilling and R. Vaught, *Borel games and the Baire property*, Trans. Amer. Math. Soc. **279** (1983), 411–428.
17. K. Schilling, *On absolutely Δ^1_2 operations*, Fund. Math. **121** (1984), 239–250.
18. W. Sierpiński, *Sur la mesurabilité des ensembles analytiques*, Spraw. Towarz. Nauk Warszaw. Wyd. III **22** (1929), 155–159.
19. E. Szpilrajn (Marczewski), *On measurability and the Baire property*, C. R. du I Congrès des Mathématiciens des Pays Slaves, Warszawa, 1929, pp. 297–303. Książnica Atlas T.N.S.W., Warszawa, 1930. [English translation available from J. C. Morgan II]
20. ——, *Sur certains invariants de l'opération (A)*, Fund. Math. **21** (1933), 229–235.
21. ——, *Sur une classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles*, Fund. Math. **24** (1935), 17–34 (esp. §2.5).
22. ——, *On absolutely measurable sets and functions*, Spraw. Towarz. Nauk. Warszaw. Wyd. III **30** (1937), 17–34 (esp. §2.2). [English translation available from J. C. Morgan II]

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