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ON BIDUALS OF C^*-TENSOR PRODUCTS

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ABSTRACT. Huruya [4] has proven that, for C^*-algebras A_1 and A_2,

$$(A_1 \otimes A_2)^{**} = A_1^{**}\overline{\otimes}A_2^{**}$$

for every A_2 if and only if A_1 is scattered. We strengthen this by proving that $(A_1 \otimes A_2)^{**} = A_1^{**}\overline{\otimes}A_2^{**}$ if and only if A_1 or A_2 is scattered. We discuss ramifications to representation theory and related questions regarding normal representations of W^*-tensor products.

1. Definitions. Let A be a C^*-algebra and σ its universal representation. We identify $\sigma(A)''$ with A^{**} and σ with the canonical embedding. If π is a representation of A, we let $\overline{\pi}$ denote the unique normal representation of A^{**} such that $\pi = \overline{\pi} \circ \sigma$. If π and θ are representations of A, we write $\pi \sim \theta$ if π and θ are quasi-equivalent, and $\pi \prec \theta$ if π is quasi-equivalent to a subrepresentation of θ (equivalently, there is a normal homomorphism ρ of $\theta(A)''$ onto $\pi(A)''$ such that $\pi = \rho \circ \theta$). Throughout this paper, A_i $(i = 1,2)$ will denote a C^*-algebra and M_i $(i = 1,2)$ a von Neumann algebra. $\odot$ (respectively, $\otimes, \overline{\otimes}$) will denote algebraic (respectively, minimal C^*-, W^*-) tensor product. We regard $A_1 \odot A_2$ as a subset of $A_1 \otimes A_2$ and $M_2 \otimes M_2$ as a subset of $M_1\overline{\otimes}M_2$, and we denote $A_1 \otimes A_2$ by A. If π is a representation of A we let π_i denote the restriction of π to A_i, so that $\pi(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2)$. We will find it convenient to denote $\pi|A_1 \odot A_2$ by $\pi_1 \times \pi_2$. Conversely, if π_1 and π_2 are commuting representations of A_1 and A_2, then $\pi_1 \times \pi_2$ is a representation of $A_1 \odot A_2$, and it extends to a representation of A if and only if it is bounded. Similarly, if ρ_1 and ρ_2 are commuting normal representations of M_1 and M_2, then $\rho_1 \times \rho_2$ extends to a normal representation of $M_1\overline{\otimes}M_2$ if and only if it is σ-weakly continuous, in which case we simply say it is normal. Note that σ_i is a representation of A_i in A^{**}, so that $\overline{\sigma}_i$ is a normal representation of A_i^{**} in A^{**}. It is not hard to check (see, e.g., [1, proof of Theorem 2]) that $\bar{\sigma}_i$ is faithful, and we find it convenient to blur the distinction between σ_i and the universal representation of A_i.

It is easy to see that if π is a representation of A, then $\overline{\pi}_1 \times \overline{\pi}_2$ is normal if and only if $\pi \prec \sigma_1 \otimes \sigma_2$, and this holds for every π if and only if $\sigma \sim \sigma_1 \otimes \sigma_2$. In this latter case we identify A^{**} with $A_1^{**}\overline{\otimes}A_2^{**}$.

DEFINITION 1.1. A_1 and A_2 are *uncorrelated* if $A^{**} = A_1^{**}\overline{\otimes}A_2^{**}$. Otherwise we say they are *correlated.*

If A_1 and A_2 are uncorrelated, then by definition $\overline{\sigma}_1 \times \overline{\sigma}_2$ is normal. If $\sigma_1 \times \sigma_2$ is merely bounded, Archbold and Batty [2] say that there is a *canonical embedding* of $A_1^{**} \otimes A_2^{**}$ in A^{**}, and they observe that this condition is satisfied if A_1 or A_2 is nuclear. On the other hand, they prove that the existence of a canonical embedding

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of $A_1^{**} \otimes A_2^{**}$ in A^{**} for every A_2 is a strictly weaker condition on A_1 than nuclearity. In particular, it is possible for there to exist a canonical embedding of $A_1^{**} \otimes A_2^{**}$ even if neither A_1 nor A_2 is nuclear.

An analogous (but nicer) relationship holds for correlation. Jensen [**5**] calls A *scattered* if A^{**} is atomic, i.e., a direct sum of type I factors. Huruya [**4**] (although he did not use the terms "uncorrelated" or "scattered") proves that A_1 and A_2 are uncorrelated if A_1 or A_2 is scattered, and morever if A_1 and A_2 are uncorrelated for every A_2 then A_1 is scattered. We will strengthen this by proving that if A_1 and A_2 are uncorrelated, then A_1 or A_2 is scattered.

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2. Main result.

LEMMA 2.1. *Let B_i be a C^*-subalgebra of A_i $(i = 1,2)$. If A_1 and A_2 are uncorrelated, then so are B_1 and B_2.*

PROOF. Let τ_i (respectively, τ) be the universal representation of B_i (respectively, $B = B_1 \otimes B_2$). Since $\tau_i \sim \sigma_i|B_i$ and $\tau \sim \sigma|B$, we have

$$\tau_1 \otimes \tau_2 \sim \sigma_1|B_1 \otimes \sigma_2|B_2 = \sigma_1 \otimes \sigma_2|B \sim \sigma|B \sim \tau. \qquad \text{Q.E.D.}$$

LEMMA 2.2. *Let π be a representation of A, and let $M_i = \pi_i(A_i)''$ $(i = 1,2)$. Further, let id_i be the identity representation of M_i $(i = 1,2)$. Then $\pi \prec \sigma_1 \otimes \sigma_2$ if and only if $\mathrm{id}_1 \times \mathrm{id}_2$ is normal.*

PROOF. We have $\pi \prec \sigma_1 \otimes \sigma_2$ if and only if $\overline{\pi}_1 \times \overline{\pi}_2$ is normal. Letting $p\colon A_1^{**} \odot A_2^{**} \to M_1 \odot M_2$ denote the quotient map, we see that $\overline{\pi}_1 \times \overline{\pi}_2 = (\mathrm{id}_1 \times \mathrm{id}_2) \circ p$, so that $\overline{\pi}_1 \times \overline{\pi}_2$ is normal if and only if $\mathrm{id}_1 \times \mathrm{id}_2$ is. Q.E.D.

THEOREM 2.3. *If A_1 and A_2 are uncorrelated, then A_1 or A_2 is scattered.*

PROOF. Suppose neither A_1 nor A_2 is scattered. By [**4**, Theorem] A_i has a commutative nonscattered C^*-subalgebra B_i. We identify B_i with $C_0(X_i)$, the space of continuous functions on a locally compact Hausdorff space X_i which vanish at infinity. Let $\overline{X}_i$ be the one-point compactification of X_i. Then $\overline{X}_i$ is not dispersed, i.e., $\overline{X}_i$ has nonempty perfect subsets [**7**].

Hence, by [**7**, Theorem], [0,1] is a continuous image of $\overline{X}_i$. Suppose the point at infinity is sent to $t \in [0,1]$ and let $I_t = [0,1]\setminus\{t\}$. Then B_i contains a C^*-subalgebra C_i isomorphic to $C_0(I_t)$. Let π_i be the representation of C_i corresponding to the usual representation on $L^2[0,1]$ by multiplication. Then $\pi_i(C_i)'' = L^\infty[0,1]$. It is straightforward to check that $\mathrm{id} \times \mathrm{id}$ on $L^\infty[0,1] \odot L^\infty[0,1]$ is not normal. Since C_i is nuclear, $\pi_1 \times \pi_2$ is bounded and so extends to a representation π of $C_1 \otimes C_2$. Let τ_i be the universal representation of C_i. By Lemma 2.2 we have $\pi \not\prec \tau_1 \otimes \tau_2$. We conclude that C_1 and C_2 are correlated, and hence by Lemma 2.1 so are A_1 and A_2. Q.E.D.

3. Normal representations of W^*-tensor products. If α is a C^*-norm on $A_1 \odot A_2$, we could define A_1 and A_2 to be *α-uncorrelated* if $(A_1 \otimes_\alpha A_2)^{**} = A_1^{**} \overline{\otimes} A_2^{**}$ and α_1-uncorrelation would imply α_2-uncorrelation if $\alpha_1 \geq \alpha_2$. In particular, the weakest such condition would be that of Definition 1.1, namely uncorrelation. However, as Theorem 2.3 shows, if A_1 and A_2 are uncorrelated, then A_1 or A_2 is nuclear, so that all C^*-norms on $A_1 \odot A_2$ agree. Therefore, α-uncorrelation would not actually depend upon α and would be equivalent to uncorrelation. Recall that representations of the maximal C^*-tensor product of A_1 and A_2 correspond to pairs of commuting representations of A_1 and A_2, and also to pairs of commuting normal representations of A_1^{**} and A_2^{**}.

DEFINITION 3.1. M_1 and M_2 are *normally uncorrelated* if whenever ρ_1 and ρ_2 are commuting normal representations of M_1 and M_2 it follows that $\rho_1 \times \rho_2$ is normal.

From the above discussion and Theorem 2.3, we get the following result.

COROLLARY 3.2. *The following are equivalent*:

(i) A_1 *and* A_2 *are uncorrelated*;

(ii) *If* π_1 *and* π_2 *are commuting representations of* A_1 *and* A_2, *then* $\overline{\pi}_1 \times \overline{\pi}_2$ *is normal*;

(iii) A_1^{**} *or* A_2^{**} *is atomic*;

(iv) A_1^{**} *and* A_2^{**} *are normally uncorrelated.*

If A_i^{**} is replaced by an arbitrary von Neumann algebra M_i ($i = 1, 2$), then in Corollary 3.2 (iii) still implies (iv). We suspect that the converse holds in general as well, although our concrete evidence is decidedly spotty.

Note that if ρ_1 and ρ_2 are commuting normal factor representations of M_1 and M_2, and if M_1 and M_2 are normally uncorrelated, then $(\rho_1(M_1) \cup \rho_2(M_2))''$ is isomorphic to $\rho_1(M_1)\overline{\otimes}\rho_2(M_2)$ (in the terminology of [**3**], $\rho_1(M_1)$ and $\rho_2(M_2)$ are *statistically independent*).

Conditions on commuting von Neumann algebras M_1 and M_2 under which $(M_1 \cup M_2)''$ is isomorphic to $M_1\overline{\otimes}M_2$ have been studied in [**6, 8, 3**], and the situation for C^*-algebras has been investigated in [**9**]. As a typical application of these results to our context we take the following from [**3**, Theorem 1]:

PROPOSITION 3.3. *If* M_1 *and* M_2 *are normally uncorrelated, and if* ρ_1 *and* ρ_2 *are commuting normal factor representations of* M_1 *and* M_2, *then there exists a type* I *factor* N *such that* $\rho_1(M_1) \otimes \mathbf{C} \cdot 1_H \subset N \subset \rho_2(M_2)'\overline{\otimes}L(H)$, *where* H *is an infinite dimensional Hilbert space.*

REFERENCES

1. R. J. Archbold, *On the centre of a tensor product of C^*-algebras* J. London Math. Soc. (2) **10** (1975), 257–262.
2. R. J. Archbold and C. J. K. Batty, *C^*-tensor norms and slice maps*, J. London Math. Soc. (2) **22** (1980), 127–138.
3. C. D'Antoni and R. Longo, *Interpolation by type* I *factors and the flip automorphism*, J. Funct. Anal. **51** (1983), 361–371.
4. T. Huruya, *A spectral characterization of a class of C^*-algebras*, Sci. Rep. Niigata Univ. Ser. A **15** (1978), 21–24.
5. H. E. Jensen, *Scattered C^*-algebras*, Math. Scand. **41** (1977). 308–314.
6. M. Nakamura, *On the direct product of finite factors*, Tôhoku Math. J. **6** (1954), 205–209.
7. A. Pełczyński and Z. Semadeni, *Spaces of continuous functions.* III: *Spaces $C(\Omega)$ for Ω without perfect subsets*, Studia Math. **18** (1959), 211–222.
8. M. Takesaki, *On the direct product of W^*-factors*, Tôhoku Math J. **10** (1958), 116–119.
9. T. Turumaru, *On the direct product of operator algebras.* IV, Tôhoku Math. J. **8** (1956), 281–285.

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