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MAXIMAL IDEALS IN SUBALGEBRAS OF $C(X)$

LOTHAR REDLIN AND SALEEM WATSON

ABSTRACT. Let X be a completely regular space, and let $A(X)$ be a subalgebra of $C(X)$ containing $C^*(X)$. We study the maximal ideals in $A(X)$ by associating a filter $\mathcal{Z}(f)$ to each $f \in A(X)$. This association extends to a one-to-one correspondence between $\mathcal{M}(A)$ (the set of maximal ideals of $A(X)$) and βX. We use the filters $\mathcal{Z}(f)$ to characterize the maximal ideals and to describe the intersection of the free maximal ideals in $A(X)$. Finally, we outline some of the applications of our results to compactifications between vX and βX.

1. Introduction. The algebra $C(X)$ of continuous real-valued functions on a completely regular space X and its subalgebra $C^*(X)$ of bounded functions have been studied extensively (see Gillman and Jerison [**3**], and Aull [**1**]). One of the interesting problems considered in [**3**] is that of characterizing the maximal ideals in these two algebras. It is a remarkable fact that the distinct problems of identifying the maximal ideals in $C(X)$ and $C^*(X)$ have a common solution—the maximal ideals are in one-to-one correspondence with the points of βX in a natural way. The methods of achieving this correspondence, however, are quite different in the two cases. In this paper we consider this problem for subalgebras $A(X)$ of $C(X)$ that contain $C^*(X)$. We show that for such algebras the maximal ideals are in one-to-one correspondence with βX. The correspondence we construct reduces to that in [**3**] for the cases of $C(X)$ and $C^*(X)$. Thus our result puts in a common setting these apparently distinct problems.

A function is invertible in $C(X)$ if it is never zero, and in $C^*(X)$ if it is bounded away from zero. In an arbitrary $A(X)$, of course, there is no such description of invertibility which is independent of the structure of the algebra. Thus in §2 we associate to each noninvertible $f \in A(X)$ a z-filter $\mathcal{Z}(f)$ that is a measure of where f is "locally" invertible in $A(X)$. This correspondence extends to one between maximal ideals of $A(X)$ and z-ultrafilters on X. In §3 we use the filters $\mathcal{Z}(f)$ to describe the intersection of the free maximal ideals in any algebra $A(X)$. Finally, our main result allows us to introduce the notion of $A(X)$-compactness of which compactness and realcompactness are special cases. In §4 we show how the Banach-Stone theorem extends to $A(X)$-compact spaces.

2. The structure space. Throughout this paper X will denote a completely regular Hausdorff space and $A(X)$ a subalgebra of $C(X)$ containing $C^*(X)$. In this section we construct the correspondence mentioned in the introduction.

A *zero set* in X is a set of the form $Z(f) = \{x \in X: f(x) = 0\}$ for some $f \in C(X)$. The complement of a zero set is a *cozero* set. $Z[X]$ will denote the

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collection of all zero sets in X. If E is a cozero set in X we will say that $f \in A(X)$ is *E-regular* if there exists $g \in A(X)$ such that $fg|_E = 1$.

LEMMA 1. *Let $f, g \in A(X)$ and let E, F be cozero sets in X.*
(a) *If f is E-regular and $F \subseteq E$, then f is F-regular.*
(b) *If f is E-regular and F-regular, then f is $E \cup F$-regular.*
(c) *If $f(x) \geq c > 0$ for all $x \in E$, then f is E-regular.*
(d) *If $0 < f(x) \leq g(x)$ for all $x \in E$ and if f is E-regular, then g is E-regular.*
(e) *If f is E-regular and g is F-regular, then fg is $E \cap F$-regular and $f^2 + g^2$ is $E \cup F$-regular.*

PROOF. (a) Obvious.
(b) Let $h, k \in A(X)$ satisfy $hf|_E = 1$ and $kf|_F = 1$. Let $w = h + k - fhk$. Then $fw|_{E\cup F} = 1$.
(c) Let $h = \max\{c, f\}$. Then $h|_E = f|_E$ and $h \geq c$. So $0 < h^{-1} \leq c^{-1}$. Hence $h^{-1} \in C^*(X) \subseteq A(X)$, and $h^{-1}f|_E = 1$.
(d) Let $h \in A(X)$ satisfy $hf|_E = 1$. For $x \in E$, $h(x) > 0$, so $h(x)g(x) \geq h(x)f(x) = 1$. Thus by (c), there exists $k \in A(X)$ such that $khg|_E = 1$.
(e) If $hf|_E = 1$ and $kg|_F = 1$, then $hkgf|_{E\cap F} = 1$. Now $f^2 + g^2 \geq f^2$, so by (d), $f^2 + g^2$ is E-regular. Similarly, it is F-regular, and so the result follows by (b).
For $f \in A(X)$, we define

$$Z(f) = \{E \in Z[X]: f \text{ is } E^c\text{-regular}\},$$

and for $S \subseteq A(X), Z[S] = \bigcup_{f\in S} Z(f)$. We recall that a *z-filter* is a nonempty collection $\mathcal{F}$ of zero sets in X such that $\mathcal{F} = \mathcal{G} \cap Z[X]$, for some filter $\mathcal{G}$ on X.

THEOREM 1. *If f is not invertible in $A(X)$, then $Z(f)$ is a z-filter on X, and conversely.*

PROOF. If f is not invertible, $\varnothing \notin Z(f)$. Moreover, if $E, F \in Z(f)$, then by Lemma 1(b), $E \cap F \in Z(f)$. If G is a zero set containing $E \in Z(f)$, then $G \in Z(f)$ by Lemma 1(a). Hence $Z(f)$ is a z-filter.
The converse is obvious.

THEOREM 2. *If I is an ideal in $A(X)$, then $Z[I]$ is a z-filter on X.*

PROOF. Clearly $\varnothing \notin Z[I]$. If $E, F \in Z[I]$, there exist $f, g \in I$ such that f is E^c-regular and g is F^c-regular. Then $f^2 + g^2 \in I$, and by Lemma 1(e), $f^2 + g^2$ is $(E \cap F)^c$-regular. Thus $E \cap F \in Z[I]$. Finally, if F is a zero set and $F \supseteq E \in Z[I]$, then $E \in Z(f)$ for some $f \in I$, and so $F \in Z(f) \subseteq Z[I]$ by Theorem 1.
Using the notation of [**3**], we write $Z^{\leftarrow}[\mathcal{F}] = \{f \in A(X): Z(f) \subseteq \mathcal{F}\}$ for the inverse of the set function Z. We will show that if $\mathcal{F}$ is a z-filter, then $Z^{\leftarrow}[\mathcal{F}]$ is an ideal in $A(X)$, giving a converse to the above theorem. We need two preliminary lemmas.

LEMMA 2. *If $f \in A(X)$, then $\lim_{Z(f)} fh = 0$ for any $h \in A(X)$.*

PROOF. We claim $\lim_{Z(f)} f = 0$. The result will follow from this claim and Lemma 1(e), since then $\lim_{Z(fh)} fh = 0$ and $Z(fh) \subseteq Z(f)$. So let $V = (-\varepsilon, \varepsilon)$ be a neighborhood of zero in $\mathbf{R}$ and let $E = f^{-1}(V)$. Clearly f is E^c-regular (Lemma 1(b) and (c)). Thus $f^{-1}(V) \in Z(f)$ and so fh converges to zero on $Z(f)$.

LEMMA 3. *Let $\mathcal{F}$ be a z-filter on X. If* $\lim_{\mathcal{F}} fh = 0$ *for all* $h \in A(X)$, *then* $\mathcal{Z}(f) \subseteq \mathcal{F}$.

PROOF. For $E \in \mathcal{Z}(f)$ we show that there is an $F \in \mathcal{F}$ such that $F \subseteq E$. Suppose not. Then $F \cap E^c \neq \varnothing$ for every $F \in \mathcal{F}$. Let $h \in A(X)$ satisfy $fh|_{E^c} = 1$. It follows that 1 is a cluster point of $\{fh(F) \colon F \in \mathcal{F}\}$, contradicting our hypothesis.

THEOREM 3. *For any z-filter $\mathcal{F}$ on X,* $I = \mathcal{Z}^{\leftarrow}[\mathcal{F}]$ *is an ideal in* $A(X)$.

PROOF. If $f \in I$ and $g \in A(X)$, then $\mathcal{Z}(fg) \subseteq \mathcal{Z}(f)$ (Lemma 1(e)), so $fg \in I$. Now if $f, g \in I$, then by Lemma 2, $\lim_{\mathcal{F}} fh = \lim_{\mathcal{F}} gh = 0$ for every $h \in A(X)$. So $\lim_{\mathcal{F}} fh + \lim_{\mathcal{F}} gh = \lim_{\mathcal{F}}(f+g)h = 0$ for all $h \in A(X)$, and hence by Lemma 3, $\mathcal{Z}(f+g) \subseteq \mathcal{F}$. Finally, we note that since $\varnothing \notin \mathcal{F}$, I consists of noninvertible elements only.

Both $\mathcal{Z}$ and $\mathcal{Z}^{\leftarrow}$ preserve inclusion and so they map maximal elements to maximal elements. Hence $\mathcal{Z}$ provides a one-to-one correspondence between βX and the set $\mathcal{M}(A)$ of maximal ideals of $A(X)$. If $\mathcal{M}(A)$ is equipped with the hull-kernel topology, then as in [**3**] in the cases of $C^*(X)$ and $C(X)$, we have the following theorem (see [**6**] for a different method of arriving at this result).

THEOREM 4. *The maximal ideal space $\mathcal{M}(A)$ of $A(X)$ equipped with the hull-kernel topology is homeomorphic to βX.*

3. Free maximal ideals. Let M^p be the maximal ideal corresponding to $p \in \beta X$ and $\mathcal{U}^p$ the z-ultrafilter on X that converges to p, so that $\mathcal{Z}(M^p) = \mathcal{U}^p$. Using our filter $\mathcal{Z}(f)$ we see immediately that for $f \in A(X)$, $f \in M^p$ if and only if $\mathcal{Z}(f) \subseteq \mathcal{U}^p$. Thus we have the following analogue of the Gelfand-Kolmogoroff theorem [**3**, Theorem 7.3] for an arbitrary $A(X)$.

THEOREM 5. *For the maximal ideals in $A(X)$, we have*

$$M^p = \{f \in A(X) \colon p \textit{ is a cluster point of } \mathcal{Z}(f) \textit{ in } \beta X\}.$$

We now describe the intersection of all the free maximal ideals in $A(X)$. An ideal I is *free* if $\bigcap Z[I] = \varnothing$, otherwise it is *fixed*. Note that a maximal ideal is free if and only if it is of the form M^p for some $p \in \beta X \backslash X$. We call a set $E \subseteq X$ *small* if every zero set contained in E is compact. Let $\mathcal{K} = \{E \in Z[X] \colon E^c \text{ is small}\}$, and let $A_K(X) = \{f \in A(X) \colon \mathcal{Z}(f) \subseteq \mathcal{K}\}$.

THEOREM 6. $A_K(X) = \bigcap\{M^p \colon p \in \beta X \backslash X\}$.

PROOF. Let $f \in A_K(X)$. If $\mathcal{U}$ is any z-ultrafilter on X such that $\mathcal{Z}(f) \not\subseteq \mathcal{U}$, then there exist disjoint zero sets $E \in \mathcal{Z}(f)$ and $F \in \mathcal{U}$. But then $F \subseteq E^c$, so F is compact and $\mathcal{U}$ is fixed. It follows that $\mathcal{Z}(f)$ is contained in every free z-ultrafilter, and so f belongs to every free maximal ideal. Conversely, if f is in every free maximal ideal, then $\mathcal{Z}(f)$ belongs to every free z-ultrafilter. Suppose $E \in \mathcal{Z}(f)$ is not in $\mathcal{K}$. Then E^c must contain a noncompact zero set F. Since $E \cup F \supseteq E \in \mathcal{Z}(f)$, $E \cup F$ belongs to every free z-ultrafilter, and hence F belongs to no free z-ultrafilter. But clearly every noncompact zero set must belong to some free z-ultrafilter. Thus E is in $\mathcal{K}$ and $f \in A_K(X)$.

We note that if X is realcompact and $A(X) = C(X)$, then $A_K(X)$ is the family of functions on X of compact support and Theorem 8.19 of [**3**] follows from our Theorem 6. If $A(X) = C^*(X)$, then $A_K(X)$ is the family of functions on X that vanish at infinity and Lemma 3.2 in [**4**] is a special case of Theorem 6.

4. A-compactness. It is well known that C^* distinguishes among compact spaces (the Banach-Stone theorem) and that C distinguishes among realcompact spaces (Hewitt's isomorphism theorem). Theorem 4 allows us to define the notion of A-compactness which will enable us to place both of these theorems in a common setting (Theorem 7).

A maximal ideal M in $A(X)$ is *real* if $A(X)/M$ is isomorphic to $\mathbf{R}$. Every fixed maximal ideal is real. If every real maximal ideal is fixed, we will say that X is $A(X)$-*compact* (or simply A-compact). With this definition, a compact space is one that is C^*-compact while a realcompact space is C-compact.

THEOREM 7. *Let X be A-compact and Y be B-compact. If $A(X)$ is isomorphic to $B(Y)$, then X is homeomorphic to Y.*

PROOF. Since X is A-compact its points correspond to the real maximal ideals of $A(X)$ under the homeomorphism described in Theorem 4. Thus we can recover X from the ring structure of $A(X)$. Since this can be done in the same way for Y, the result follows.

Although the converse of the above theorem is trivial if A and B are C or C^*, in this more general setting the converse is not even true. For a given X there can exist nonisomorphic algebras $A(X)$ and $B(X)$ for which X is both A-compact and B-compact. For example, let $H(\mathbf{N})$ be the algebra of sequences which occur as the coefficients of the Taylor series representation of functions holomorphic on the open unit disc. Then $\mathbf{N}$ is both H-compact (see [**2**]) and C-compact, but $H(\mathbf{N})$ is obviously not isomorphic to $C(\mathbf{N})$. Indeed, it is clear from the definition that if X is A-compact and $B(X) \supseteq A(X)$, then X is B-compact. This raises the question: Does there exist in some sense a "minimal" algebra $A_m(X)$ for which X is A_m-compact, at least up to isomorphism?

We conclude by noting that another characterization of A-compactness follows from Mandelker [**5**]. We call a family $\mathcal{S}$ of closed sets in X A-stable if every $f \in A(X)$ is bounded on some member of $\mathcal{S}$. Then one can show (as in [**5**]) that a space is A-compact if and only if every A-stable family of closed sets with the finite intersection property has nonempty intersection.

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PENNSYLVANIA STATE UNIVERSITY, OGONTZ CAMPUS, ABINGTON, PENNSYLVANIA 19001

PENNSYLVANIA STATE UNIVERSITY, DELAWARE COUNTY CAMPUS, MEDIA, PENNSYLVANIA 19063