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PRIMES DIVIDING CHARACTER DEGREES AND CHARACTER ORBIT SIZES

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ABSTRACT. We consider an abelian group A which acts faithfully and coprimely on a solvable group G. We show that some A-orbit on $\mathrm{Irr}(G)$ must have cardinality divisible by almost half the primes in $\pi(A)$. As a corollary, we improve a recent result of I. M. Isaacs concerning the maximum number of primes dividing any one character degree of a solvable group.

A recent result of I. M. Isaacs [**5**, Corollary 4.3] relates the maximum number of primes dividing any one irreducible character degree of a solvable group G to the number of primes dividing all the character degrees of G taken together. Here we considerably strengthen the bound in [**5**, Corollary 4.3] by proving a result on character orbit sizes in coprime actions.

We consider an abelian group A which acts faithfully and coprimely on a solvable group G. We show that some A-orbit on $\mathrm{Irr}(G)$ must have cardinality divisible by almost half the primes in $\pi(A)$. Our approach roughly parallels that of [**6**]. This paper and [**5**] contain new applications of the results and methods of [**6, 7, 1, 2**].

I would like to thank I. M. Isaacs for bringing this problem to my attention.

Our notation is largely standard. If a group G acts on a set Ω and $\omega \in \Omega$, we denote by $\mathrm{Orb}_G(\omega)$ the G-orbit of ω. If $G \triangleleft H$ and H also acts on Ω, we say that $h \in H$ moves $\mathrm{Orb}_G(\omega)$ if $\omega^h \notin \mathrm{Orb}_G(\omega)$. All groups considered in this paper are finite and solvable.

We now state our main results.

THEOREM 1. *Let A act faithfully on G with $(|A|, |G|) = 1$ and A abelian. Then* $|\pi(A)| \leqslant 2|\pi(\mathrm{Orb}_A(\chi))| + 8$ *for some* $\chi \in \mathrm{Irr}(G)$.

COROLLARY 1. *Let G be solvable. Let* $s = \max\{|\pi(\chi(1))| \colon \chi \in \mathrm{Irr}(G)\}$ *and let* $\rho = |\pi(\prod_{\chi \in \mathrm{Irr}(G)} \chi(1))|$. *Then* $\rho \leqslant s^2 + 10s$.

PROOF. Using Theorem 1 above, we get a stronger version of [**5**, Theorem 4.1] in which part (a) is replaced by $|\rho(G) - \rho(N) - \sigma| \leqslant 2(s - |\sigma|) + 8$. This leads to a stronger version of [**5**, Corollary 4.3] in which $|\rho(G)| \leqslant s + \sum_{i=0}^{s-1}(2(s-i) + 8) = s^2 + 10s$. Since $|\rho(G)|$ in [**5**] is called ρ in our paper, this completes the proof.

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The next proposition shows that the bound in Theorem 1 is close to best possible.

PROPOSITION 1. *Let m be a positive integer. There exist groups A and G satisfying the hypotheses of Theorem 1 with $|\pi(A)| = 2m$ and $\pi(\mathrm{Orb}_A(\chi)) \leqslant m$ for all $\chi \in \mathrm{Irr}(G)$.*

PROOF. For $1 \leqslant i \leqslant m$, choose odd primes p_i and q_i subject to the following conditions. Let e_i and f_i denote the order of 2 mod p_i and q_i respectively.

(1) $\min(p_i, q_i) > 2^{p_{i-1}q_{i-1}}$ for $i > 1$,

(2) $q_i \equiv 2 \pmod{e_1 f_1 e_2 f_2 \cdots e_{i-1} f_{i-1}}$ for $i > 1$,

(3) $f_i < e_i$ for all i.

For each i set $n_i = f_i p_i$. Clearly $2^{f_i} - 1 \equiv 0 \bmod q_i$. Since

$$(2^{n_i} - 1)/(2^{f_i} - 1) \equiv 1 + 2^{f_i} + \cdots + 2^{(p_i - 1)f_i},$$

it follows that $(2^{n_i} - 1)/(2^{f_i} - 1) \equiv p_i \not\equiv 0 \bmod q_i$. By (3) above, $2^{n_i} - 1 = 2^{f_i p_i} - 1 \equiv 2^{f_i} - 1 \not\equiv 0 \bmod p_i$.

Let V_i be elementary abelian of order 2^{n_i}. Let C_i be cyclic of order $2^{n_i} - 1$. Let P_i be cyclic of order p_i and let P_i act on $C_i \cong \mathrm{GF}(2^{n_i} - 1)^\times$ as the subgroup of order p_i in $\mathrm{Gal}(\mathrm{GF}(2^{n_i})/\mathrm{GF}(2))$. Let $H_i = (P_i C_i)V_i$ be the corresponding subgroup of the affine semilinear group over $\mathrm{GF}(2^{n_i})$. Let Q_i be the Sylow q_i-subgroup of C_i. The preceding paragraph implies that $[P_i, Q_i] = 1$ and $Q_i > 1$. Let $C_i = Q_i \times R_i$. Let $G_i = R_i V_i$ and $A_i = P_i Q_i$. Then A_i acts faithfully on G_i and the preceding paragraph shows that $(|A_i|, |G_i|) = 1$.

Let $\chi \in \mathrm{Irr}(H_i)$ and let λ be an irreducible constituent of χ_{V_i}. If $\lambda = 1$, then $\chi(1)$ divides p_i. If $\lambda \neq 1$, then $|I_{H_i}(\lambda){:}\ V_i| = p_i$, and every character in $\mathrm{Irr}(I_{H_i}(\lambda)|\lambda)$ has degree 1 (see [**4**, 6.17 and 6.20]). Hence $\chi(1) = |Q_i||R_i|$. In either case, at most one prime in $\pi(A_i)$ divides $\chi(1)$.

Let $\psi \in \mathrm{Irr}(G_i)$. Since $G_i \triangleleft H_i$, $|\mathrm{Orb}_{A_i}(\psi)|$ divides an irreducible character degree of H_i. Hence $|\pi(\mathrm{Orb}_{A_i}(\psi))| \leqslant 1$.

Now let $G = G_1 \times \cdots \times G_m$ and let $A = A_1 \times \cdots \times A_m$ act componentwise on G. Clearly A acts faithfully on G. To show that $(|A|, |G|) = 1$, it suffices, by induction, to show that $(|A_m|, |G_i|) = (|A_i|, |G_m|) = 1$ for $i \leqslant m$. We already know that $(|A_m|, |G_m|) = 1$. Suppose $i < m$. If $(|A_m|, |G_i|) > 1$, then $(|A_m|, |C_i|) = (|A_m|, 2^{p_i f_i} - 1) > 1$. Since f_i divides $q_i - 1$, this contradicts condition (1) in the first paragraph. If $(|A_i|, |G_m|) > 1$ for $i < m$, then $(|A_i|, |C_m|) = (|A_i|, 2^{p_m f_m} - 1) > 1$. Then $\mathrm{ord}_r(2) \mid p_m f_m$, where $r = p_i$ or $r = q_i$. Since $\mathrm{ord}_r(2)$ divides $r - 1$ and $r < p_m$, we have $\mathrm{ord}_r(2) \mid f_m$ and so $\mathrm{ord}_r(2) \mid (q_m - 1)$, contrary to condition (2) in the first paragraph. Hence A and G satisfy the hypothesis of Theorem 1.

Let $\psi = \psi_1 \times \cdots \times \psi_m$ be an arbitrary character in $\mathrm{Irr}(G)$. As above, we may choose $S_i \in \{P_i, Q_i\}$ so that S_i fixes ψ_i. Then $S_1 \times \cdots \times S_m$ fixes ψ, so $|\pi(\mathrm{Orb}_A(\psi))| \leqslant m$. This completes the proof.

We proceed to prove Theorem 1. We will use the proof of [**6**, Theorem 3.3] as a rough guide in the proof of Proposition 3 below.

LEMMA 1. *Let A act on G with A abelian and $(|A|, |G|) = 1$. Let A_p denote the p-Sylow subgroup of A. Let N and M be A-invariant normal subgroups of G, with $[A_p, G] \leqslant M$ and $N \leqslant M$. Let λ be an irreducible character of N and suppose A_p*

moves $\mathrm{Orb}_M(\lambda)$. *Then* A_p *moves* $\mathrm{Orb}_G(\lambda)$. *Similarly, if* $v \in N$ *and* A_p *moves* $\mathrm{Orb}_M(v)$, *then* A_p *moves* $\mathrm{Orb}_G(v)$.

PROOF. Suppose A_p stabilizes $\mathrm{Orb}_G(\lambda)$. Then the semidirect product A_pG acts on $\mathrm{Orb}_G(\lambda)$ with G acting transitively. By Glauberman's Lemma [**4**, Lemma 13.8], A_p fixes some $\psi \in \mathrm{Orb}_G(\lambda)$. Then A_p^g fixes λ for some $g \in G$. Since $G = C_G(A_p)[A_p, G] = C_G(A_p)M$, we may assume that $g \in M$. Hence A_p stabilizes $\mathrm{Orb}_M(\lambda)$.

The second assertion is proved similarly.

LEMMA 2. *Let* $G \neq 1$ *be solvable with every normal abelian subgroup cyclic. Let* $p_1, \dots, p_n$ *be the distinct prime divisors of* $|F(G)|$ *and let* $Z \leqslant Z(F(G))$ *with* $|Z| = p_1 \cdots p_n$. *Let* $D = C_G(Z)$. *Then there exist* $E, T \lhd G$ *with*

(i) $ET = F(G)$ *and* $E \cap T = Z$.

(ii) *Each Sylow subgroup of* T *is cyclic, dihedral, semidihedral or quaternion.*

(iii) T *has a cyclic subgroup* U *with* $|T : U| \leqslant 2$ *and* $U \lhd G$.

(iv) *Each Sylow subgroup of* E *is cyclic of prime order or extraspecial of prime exponent or exponent* 4.

(v) G *is nilpotent if and only if* $G = T$.

(vi) $T = C_G(E)$ *and* $F(G) = C_D(E/Z)$.

(vii) *Each Sylow subgroup of* E/Z *is elementary abelian and is a completely reducible* $D/F(G)$*-module.*

PROOF. This is [**7**, Corollary 2.4].

LEMMA 3. *Let* G, E, U, *and* Z *be as in Lemma* 2. *Let* V *be a faithful* $F[EU]$*-module for a finite field* F. *Let* $W \neq 0$ *be an irreducible* U*-submodule of* V *and let* $e = |E : Z|^{1/2}$. *Then* $\dim V = me \dim W$ *for an integer* m.

PROOF. This is [**7**, Lemma 2.5].

LEMMA 4. *Let* $E \lhd H$ *with* $|H : E| = p$ *and* $p \nmid |E|$. *Let* $Z = Z(E)$, $P \in \mathrm{Syl}_p(H)$, *and let* V *be a finite-dimensional* $F[H]$*-module for a field* F. *Assume that* E/Z *is an abelian* q*-group for a prime* q, $P \nleqslant C_H(E)$, *and* V_E *is a faithful, completely reducible and homogeneous module. Then* $\dim C_V(P) \leqslant (\dim V)/2$ *if* p *is odd.*

PROOF. This is part of [**6**, Lemma 1.7].

LEMMA 5. *Let* $V \neq 0$ *be a faithful and completely reducible* $F[G]$*-module for a field* F *and a solvable group* G. *Then* $|G| \leqslant |V|^{9/4}$.

PROOF. This is a slightly weaker version of [**7**, Theorem 3.1].

PROPOSITION 2. *Let* A *act on* G *with* $(|A|, |G|) = 1$ *and* A *cyclic of squarefree order. Suppose that* $[A_p, G/F(G)]$ *is a nonidentity abelian group for all* p *in* $\pi(A)$. *Then* A *has a faithful orbit on* $\mathrm{Irr}(G)$.

PROOF. Let $H = G/F(G)$. Then $[A, H] = \prod_{p \in \pi(A)}[A_p, H]$ is contained in $F(H)$. Let $W = [A, H]/\Phi([A, H])$, so that W is a direct product of elementary abelian q-groups for primes q dividing $|H|$. Write $W = W_1 \times \cdots \times W_k$, each W_i an irreducible A-module. For $1 \leqslant i \leqslant k$, let $1 \neq \lambda_i \in \mathrm{Irr}(W_i)$. Let $\lambda = \lambda_1 \times \cdots \times \lambda_k$. Since

$(|A|, |H|) = 1$, A acts faithfully on $[A, H]$ and hence on W. For each i, W_i is a faithful irreducible module for the cyclic group $A/C_A(W_i)$. Hence λ_i is moved by A_p for every $p \in \pi(A/C_A(W_i))$. Thus λ lies in a faithful A-orbit on $\mathrm{Irr}(W) \leqslant \mathrm{Irr}([A, H])$. We now apply Lemma 1 with $A, H, [A, H], [A, H]$ in place of A, G, M, N and conclude that $\mathrm{Orb}_G(\lambda)$ is moved by A_p for every $p \in \pi(A)$. Let $\chi \in \mathrm{Irr}(H \mid \lambda)$. Then χ lies in a faithful A-orbit on $\mathrm{Irr}(H) \leqslant \mathrm{Irr}(G)$.

PROPOSITION 3. *Let* $\pi_0 = \{2, 3, 5, 7, 11, 13, 17, 31\}$. *Let A be cyclic of squarefree π_0'-order. Let A act on G with* $(|A|, |G|) = 1$ *and* $[A_p, G]$ *nonabelian for all* $p \in \pi(A)$. *Let V be an abelian group which is a direct product of completely reducible AG-modules over various finite fields. Suppose* $(|A|, |V|) = 1$ *and AG acts faithfully on V. Then there exists $v \in V$ such that* $\mathrm{Orb}_G(v)$ *is moved by every* A_p.

PROOF. We proceed by induction on $|G| + |V|$. Set $\pi(A) = \pi$.

First suppose V is not an irreducible AG-module. Write $V = V_1 \times \cdots \times V_k$, with each V_i an irreducible AG-module. Let $\bar{G}_i = G/C_G(V_i)$ for $1 \leqslant i \leqslant k$. Let $\pi_i = \{ p \in \pi$: $[A_p, \bar{G}_i]$ is nonabelian$\}$. For each $p \in \pi$, $[A_p, G]$ is isomorphic to a subgroup of $[A_p, \bar{G}_1] \times \cdots \times [A_p, \bar{G}_k]$. Hence $[A_p, \bar{G}_i]$ is nonabelian for some i, and so $\pi = \pi_1 \cup \cdots \cup \pi_k$. We apply the induction hypothesis to $\bar{G}_i$ with $\prod_{p \in \pi_i} A_p$, $\bar{G}_i$, V_i in place of A, G, V. We obtain $v_i \in V_i$ such that $\mathrm{Orb}_G(v_i)$ is moved by A_p for all $p \in \pi_i$. Then $\mathrm{Orb}_G(v_1, \ldots, v_k)$ is moved by A_p for all $p \in \pi$.

We now assume that V is an irreducible AG-module. We may apply Lemma 1 with A, GV, $[A, G]V, V$ in place of A, G, M, N and conclude that it suffices to find $v \in V$ such that $\mathrm{Orb}_{[A,G]}(v)$ is moved by A_p for all $p \in \pi$. By the inductive hypothesis we may then assume that $G = [A, G]$. It follows that $O^{\pi'}(AG) = AG$, since otherwise a proper factor group of G would be centralized by A, contrary to $G = [A, G]$.

Suppose that V is imprimitive. Let $V = V_1 \oplus \cdots \oplus V_t$ be an imprimitivity decomposition for the action of AG on V. We may partition $\{1, \ldots, t\}$ into blocks B_j, $1 \leqslant j \leqslant s$, and set $U_j = \sum_{i \in B_j} V_i$, so that AG permutes the set $\{U_1, \ldots, U_s\}$ primitively. Let C be the kernel of the permutation action of AG on the U_j. Since $AG = O^{\pi'}(AG)$, we have $A \nleqslant C$. By [**1**, Theorem 1] we may choose $S \leqslant \{1, 2, \ldots, s\}$ so that the stabilizer in AG of $\sum_{j \in S} U_j$ is C. Let $U = \sum_{j \in S} U_j$. Let $\pi_1 = \{ p \in \pi$: $A_p \leqslant C\}$.

Let $A_1 = \prod_{p \in \pi_1} A_p$ so that $C = A_1(C \cap G)$. All the irreducible constituents of $V_{C \cap G}$ are AG-conjugate. Thus if K_j denotes the kernel of C on U_j, then $\bigcap_{x \in AG} K_j^x = 1$ for each $j \in S$. Let $p \in \pi_1$. Since all p-Sylow subgroups of C are conjugate under $G \cap C$, it follows that $[A_p, G \cap C]$ char $G \cap C$, and so $[A_p, G \cap C]'$ char $G \cap C$. The last two sentences imply that $[A_p, G \cap C]' \nleqslant K_j$ for any $j \in S$. Hence $A_p \nleqslant K_j$ for $p \in \pi_1$ and $j \in S$. Since $K_j \lhd C$, it follows that $K_j \leqslant G \cap C$ and $[A_p, (G \cap C)/K_j]$ is nonabelian for $p \in \pi_1$ and $j \in S$.

For $j \in S$, we may now apply the inductive hypothesis with A_1, $(G \cap C)/K_j$, U_j in place of A, G, V. We obtain $u_j \in U_j$ such that $\mathrm{Orb}_{G \cap C}(u_j)$ is moved by A_p for all $p \in \pi_1$. Let $u = \sum_{j \in S} u_j$. If $p \in \pi - \pi_1$ and $P \in \mathrm{Syl}_p(AG)$, then the choice of S insures that P does not centralize u. If $P \in \mathrm{Syl}_p(AG)$ and $p \in \pi_1$, then the definition of u_j implies that P does not centralize u. For every $p \in \pi$, the last two sentences show that A_p fixes no element in $\mathrm{Orb}_G(u)$. If A_p stabilized $\mathrm{Orb}_G(u)$, then Glauberman's Lemma applied to the action of A_pG on $\mathrm{Orb}_G(u)$ would yield a contradiction. Hence A_p moves $\mathrm{Orb}_G(u)$ as desired.

We may now assume that V is a primitive AG-module. If $F(AG) \nleq G$, then some $A_p \leqslant F(AG)$, and so $A_p \triangleleft AG$, contrary to $[A_p, G] \neq 1$. Hence $F(AG) \leqslant G$, and so $F(AG) = F(G)$. Set $F(AG) = F$. Now AG can play the role of "G" in Lemma 2. Let T, $U \leqslant AG$ be as in the conclusion of Lemma 2. Suppose $T \neq U$. Then every $2'$-element of AG centralizes $O_2(T)$, so $AG/C_{AG}(O_2(T))$ is a nonidentity 2-group, contradicting $O^{\pi'}(AG) = AG$. Thus $T = U$ is cyclic. Let Z, D, and E be as in Lemma 2, so that $F = C_D(E/Z)$ and each Sylow subgroup of E/Z is a completely reducible D/F-module.

Fix $p \in \pi$. Since AG/D is abelian, $[A_p, G] \leqslant D \cap G$. Thus $[A_p, G] = [A_p, A_p, G] = [A_p, D \cap G]$. Suppose $[A_p, E/Z] = 1$. Since D and $C_{AG}(E/Z)$ are normal in AG, we have $[A_p, G] = [A_p, D \cap G] \leqslant C_D(E/Z) = F$. Hence

$$[A_p, G] = [A_p, A_p, G] = [A_p, F] = [A_p, E][A_p, T] \leqslant ZT = T,$$

contrary to the hypotheses of Proposition 3. Hence $[A_p, E/Z] \neq 1$. Let E_1 be a Sylow subgroup of E with $[A_p, E_1/E_1 \cap Z] \neq 1$. We apply Lemma 4 to A_pE_1, E_1, A_p, V in place of H, E, P, V. We conclude that $|C_V(A_p)| \leqslant |V|^{1/2}$.

Let Y be an irreducible F-submodule of V. By Lemma 3, $|Y| = |W|^{me}$, where $e^2 = |E:Z|$ and m is a positive integer. Moreover W is a faithful irreducible T-submodule of Y, so that $|T|$ divides $|W| - 1$.

Now $|G \cap D| = |G \cap D : F||F|$. By Lemmas 2 and 5, and an obvious subdirect product argument, $|D:F| \leqslant |E:Z|^{9/4} = e^{9/2}$. We have $|G \cap D| \leqslant e^{9/2}e^2|T| = |T|e^{13/2}$. Since $[A_p, G] = [A_p, G \cap D]$, we have $O^{p'}(AG) \leqslant A_p(G \cap D)$, so $|\mathrm{Syl}_p(AG)| \leqslant |G \cap D| \leqslant e^{13/2}|T|$. Hence

$$\sum_{P \in \mathrm{Syl}_p(AG)} |C_V(P)| \leqslant |T|e^{13/2}|C_V(A_p)| \leqslant |T|e^{13/2}|V|^{1/2}.$$

We will show that the following inequality holds:

$$(*) \qquad \sum_{P \in \mathrm{Syl}_p(AG)} |C_V(P)| \leqslant p^{-2}|V|.$$

Suppose $(*)$ is false, so that $p^2|T|e^{13/2} > |V|^{1/2} \geqslant |W|^{e/2}$.

If $[A_p, Z] \neq 1$, then p divides $|\mathrm{Aut}\, Z|$ and so $p \mid (s-1)$ for some prime divisor s of $|Z|$. Since $Z = T \cap E$, we have $s \mid e$ and $s \leqslant |T| < |W|$. Since $p > 17$ and $p \mid (s-1)$, it follows that $s \geqslant 47$. Since $p < s \leqslant |T| < |W|$, we have $|W|^3e^{13/2} > |W|^{e/2}$, so that $e^{13/2} > |W|^{(e/2)-3} > 48^{(e/2)-3}$. Hence $e < 20$, contrary to $s \geqslant 47$ and $s \mid e$.

Thus we assume $[A_p, Z] = 1$. Let $e = \prod_i q_i^{n_i}$ for distinct primes q_i. Since $[A_p, E/Z] \neq 1$ and $A_p \leqslant D$, p divides $|\mathrm{Sp}(2n_i, q_i)|$ for some i. Hence $p \mid q_i^{2m_i} - 1$ for some m_i with $1 \leqslant m_i \leqslant n_i$. Thus $p \mid q_i^{m_i} + 1$ or $p \mid q_i^{m_i} - 1$, where $q_i^{m_i} \mid e$. It follows that $p \leqslant e + 1$. Since $|T| < |W|$, we have $(e+1)^2e^{13/2} > |W|^{(e/2)-1}$. Since $|W| \geqslant 3$, we have $e < 70$.

If $m_i > 1$, our hypothesis that $p \notin \pi_0$ implies that $q_i^{m_i} > 70$. Hence $e > 70$, a contradiction. Thus $m_i = 1$ and $p \notin \pi_0$ implies that $q_i \geqslant 37$. Since q_i divides $|T|$ and $|T|$ divides $|W| - 1$ and $|W|$ is a prime power, we must have $|W| \geqslant 83$. Hence $(e+1)^2 e^{13/2} > 83^{(e/2)-1}$. As above, this implies that $e < 20$, contrary to $q_i | e$ and $q_i \geqslant 37$.

We conclude that (*) holds for each $p \in \pi$. Since $\sum_{p \in \pi} p^{-2}$ is less than 1, it follows that there exists $v \in V$ such that v is centralized by no p-Sylow subgroup of AG for any $p \in \pi$. Then no A_p fixes any element in $\mathrm{Orb}_G(v)$. By Glauberman's Lemma, each A_p moves $\mathrm{Orb}_G(v)$.

PROPOSITION 4. *Let A be cyclic of squarefree π_0'-order. Let N be an A-invariant normal abelian subgroup of G which is a direct product of completely reducible AG-modules. Suppose $N = C_{AG}(N)$. For each $p \in \pi(A)$, suppose that $[A_p, G/N]$ is either nonabelian or trivial. Then A has a faithful orbit on* $\mathrm{Irr}(G)$.

PROOF. Let $V = \mathrm{Irr}(N)$, $\pi = \pi(A)$, $\pi_1 = \{p \in \pi\colon [A_p, G/N]$ is nonabelian$\}$, and $\pi_2 = \pi - \pi_1$. Let A_1 be the Hall π_1-subgroup of A.

Then A_1, G/N, and N satisfy the hypotheses of Proposition 3. Hence we may choose $v \in V$ so that $\mathrm{Orb}_G(v)$ is moved by A_p for all $p \in \pi_1$.

Write $V = V_1 \times \cdots \times V_k$, a direct product of irreducible AG-modules. We may assume that each component v_i of v is not 1. For each $p \in \pi_2$, we may choose $i \in \{1, \ldots, k\}$ such that V_i is not centralized by A_p. Since $[A_p, G] \leqslant N = C_{AG}(V)$, the centralizer in V_i of A_p is an AG-submodule of V_i, and hence is trivial. Thus A_p moves v_i, and so A_p moves v.

We think of v as a linear character λ of N. Then $\mathrm{Orb}_N(\lambda) = \{\lambda\}$ is not A_p-invariant for any $p \in \pi_2$. By Lemma 1 with A, G, N, N in place of A, G, M, N, we conclude that $\mathrm{Orb}_G(\lambda)$ is moved by A_p for all $p \in \pi_2$. By the second paragraph, $\mathrm{Orb}_G(\lambda)$ is moved by A_p for all $p \in \pi$. Hence if $\chi \in \mathrm{Irr}(G|\lambda)$, then χ lies in a faithful A-orbit.

PROOF OF THEOREM 1. We may assume A is cyclic of squarefree order. The hypotheses of Theorem 1 imply that $F(AG) = F(G)$. Since $G = [A_p, G]C_G(A_p)$ for each $p \in \pi(A)$, it follows that A acts faithfully on $G/\Phi(AG)$. Thus we may assume that $\Phi(AG) = 1$ and hence that $F(G) = F(AG)$ is a direct product of completely reducible AG-modules (see [**3**, III, Satz 4.5]).

Partition $\pi(A) = \pi$ as follows. Let $\pi_1 = \pi \cap \pi_0$, $\pi_2 = \{p \in \pi\colon\ p \notin \pi_0$ and $[A_p, G/F(G)]$ is abelian and nontrivial$\}$, $\pi_3 = \pi - \pi_1 - \pi_2$, and π_4 be the larger of π_2 and π_3. Let A_4 be the Hall π_4-subgroup of A. By Proposition 3 or Proposition 4 applied to A_4 and G, we may conclude that A_4 has a faithful orbit on $\mathrm{Irr}(G)$. Since $|\pi(A)| \leqslant 2|\pi(A_4)| + 8$, this completes the proof.

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