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ITERATIONS OF HARDY-LITTLEWOOD MAXIMAL FUNCTIONS

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ABSTRACT. It is shown that the iteration of two maximal functions is essentially no larger than the inner maximal function.

1. For a measurable $f\colon \mathbf{R}^n \to \mathbf{R}$ we let $\lambda_f(y) = |\{x\colon |f(x)| > y\}|$, $y > 0$, and define

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty \lambda_f(y)^{q/p}\, d(y^q)\right)^{1/q}, & 1 \leqslant q < \infty, \\ \sup_{y>0} y\lambda_f(y)^{1/p}, & q = \infty. \end{cases}$$

This is the familiar Lorentz space $L(p,q)$ "norm" of f **[4]** written in terms of the distribution function λ_f rather than the nonincreasing rearrangement of f. It is well known **[4]** that $\|f\|_{p,p} = \|f\|_p$, the L^p-norm of f, $\|f\|_{p,q_2} \leqslant \|f\|_{p,q_1}$, $q_2 \geqslant q_1$, and $\|f+g\|_{pq} \leqslant 2(\|f\|_{pq} + \|g\|_{p,q})$. This paper is concerned with maximal operators of the form

$$M_{p,q}f(x) = \sup_{x\in Q} \frac{\|f\chi_Q\|_{p,q}}{|Q|^{1/p}},$$

where the sup is extended over all cubes Q with $x \in Q$. In particular, we will study iterations $M_{p,q}M_{r,s}f$ and we will obtain inequalities of the type $M_{p,q}M_{r,s}f(x) \leqslant CM_{r,s}f(x)$ under certain restrictions on the indices. Special cases of this have been studied extensively. For example, Coifman and Rochberg **[3]** proved $MM_rf \leqslant CM_rf$, $1 < r < \infty$, where $M = M_{1,1}$, the usual Hardy-Littlewood maximal function, and $M_r = M_{r,r}$, thus showing that $M_rf \in A_1$, Muckenhoupt's weight class A_1 **[6]**. More recently, Bruna and Korenblum **[1, 2]** have established $M_{1,\infty}Mf(x) \leqslant CMf(x)$, and among other results, used this to show that $M_{1,\infty}T^*f \leqslant C(T^*f + Mf)$, where T^* is the maximal Calderón-Zygmund singular integral operator.

The operator $M_{p,q}$ has also been studied by Stein **[7]** who for $q \leqslant p$ established the weak-type inequality $|\{x\colon M_{p,q}f(x) > y\}| \leqslant C(\|f\|_{p,q}/y)^p$. A slightly more general result of this kind can also be found in **[5]**.

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2. We will restrict ourselves to dyadic maximal functions, and at the end of this paper we will make a remark about the nondyadic case.

Let Δ be the collection of dyadic cubes Q in $\mathbf{R}^n$ (see **[8]**), and define $M_{p,q}f(x) \equiv M^{\Delta}_{p,q}f(x)$ as before except that the sup is taken over all $Q \in \Delta$ with $x \in Q$.

THEOREM 1. *Let* $1 \leqslant p < r < \infty$, $1 \leqslant s \leqslant r$, *and* $1 \leqslant q < \infty$. *Then* $M_{p,q}M_{r,s}f(x) \leqslant CM_{r,s}f(x)$, $x \in \mathbf{R}^n$, *where* $C = 2^{1/q}(p/(r-p))^{p/rq}$.

PROOF. Fix $x_0 \in \mathbf{R}^n$, and let $\tau < M_{p,q}M_{r,s}f(x_0)$. Then there exists $x_0 \in Q \in \Delta$ such that

$$\tau^q < \frac{1}{|Q|^{q/p}} \int_0^\infty \lambda_{(M_{r,s}f)\cdot\chi_q}(y)^{q/p}\, d(y^q) = \frac{1}{|Q|^{q/p}}\left(\int_0^A + \int_A^\infty\right).$$

We will estimate both integrals. Since $\lambda_{(M_{r,s}f)\cdot\chi_Q}(y) \leqslant |Q|$, the first integral $(1/|Q|^{q/p})\int_0^A \leqslant A^q$.

To estimate the second integral, let $E_y = \{x\colon\ M_{r,s}f(x) > y\}$. Then $E_y = \bigcup Q_j$, $Q_j \in \Delta$, $Q_i \cap Q_j = \varnothing$, $i \neq j$, and

$$y < \frac{1}{|Q_j|^{1/r}}\left(\int_0^\infty \lambda_{f\chi_{Q_j}}(t)^{s/r}\, d(t^s)\right)^{1/s}.$$

Hence

$$\frac{|E_y \cap Q|}{|Q|} = \frac{1}{|Q|}\sum |Q_j \cap Q|.$$

Since we deal with the dyadic case, if $Q_j \cap Q \neq \varnothing$, then either $Q_j \subset Q$ or $Q_j \supset Q$. In the second case let $Q = Q_j$ without decreasing $|E_y \cap Q|/|Q|$. Thus we may suppose that $E_y \cap Q = \bigcup_{Q_j \subset Q} Q_j$, and hence, since

$$|Q_j| < \frac{1}{y^r}\left(\int_0^\infty \lambda_{f\chi_{Q_j}}(t)^{s/r}\, d(t^s)\right)^{r/s},$$

we get

$$\begin{aligned} |E_y \cap Q| &\leqslant \sum_{Q_j \subset Q} |Q_j| \leqslant \frac{1}{y^r} \sum_{Q_j \subset Q} (\ )^{r/s} \\ &\leqslant \frac{1}{y^r}\left\{\int_0^\infty \left[\sum_j \lambda_{f\cdot\chi_{Q_j}}(t)\right]^{s/r} d(t^s)\right\}^{r/s} \qquad (\text{since } r \geqslant s) \\ &\leqslant \frac{1}{y^r}\left\{\int_0^\infty \lambda_{f\chi_Q}(t)^{s/r}\, d(t^s)\right\}^{r/s}. \end{aligned}$$

From this we obtain

$$\begin{aligned} \frac{\lambda_{(M_{r,s}f)\chi_q}(y)}{|Q|} = \frac{|E_y \cap Q|}{|Q|} &\leqslant \frac{1}{|Q|}\frac{1}{y^r}\left(\int_0^\infty \lambda_{f\chi_Q}(y)^{s/r}\, d(t^s)\right)^{r/s} \\ &\leqslant \frac{1}{y^r} M_{r,s}f(x_0)^r. \end{aligned}$$

Up to here we only used the restriction $r \geqslant s$. Since $r > p$ we get

$$\frac{1}{|Q|^{q/p}} \int_A^\infty \leqslant M_{r,s} f(x_0)^{qr/p} q \int_A^\infty y^{q(1-r/p)-1}\,dy = M_{r,s} f(x_0)^{qr/p} \frac{p}{r-p} A^{q-qr/p}.$$

We combine the two estimates and obtain

$$\tau^q \leqslant A^q + \frac{p}{r-p} M_{r,s} f(x_0)^{qr/p} A^{q-qr/p}.$$

If we choose A so that the two terms on the right are equal, i.e., $A = M_{r,s} f(x_0)(p/(r-p))^{p/qr}$, we obtain $\tau^q \leqslant 2A^q$ or $\tau \leqslant 2^{1/q} A$, and the proof is complete.

3. This paragraph is concerned with the case $q = \infty$ of Theorem 1.

THEOREM 2. *Let* $1 \leqslant p \leqslant r < \infty$, $1 \leqslant s \leqslant r$. *Then* $M_{p,\infty} M_{r,s} f(x) \leqslant M_{r,s} f(x)$, $x \in \mathbf{R}^n$.

PROOF. If $r > p$, we get from Theorem 1 that

$$M_{p,\infty} M_{r,s} f(x) \leqslant M_{p,q} M_{r,s} f(x) \leqslant c_q M_{r,s} f(x)$$

for $q < \infty$, where $c_q = 2^{1/q}(p/(r-p))^{p/rq}$. Hence $c_q \to 1$ as $q \to \infty$, and this is the $r > p$ case of the theorem.

We assume now $r = p$. Fix $x_0 \in \mathbf{R}^n$ and let $\tau < M_{p,\infty} M_{p,s} f(x_0)$. Then there is $Q \in \Delta$, $x_0 \in Q$, and there is $y > 0$ such that

$$\tau < \frac{1}{|Q|^{1/p}} y \lambda_{(M_{p,s} f)\chi_Q}(y)^{l/p}.$$

As in Theorem 1, $\lambda_{(M_{p,s} f)\chi_Q}(y)/|Q| \leqslant (1/y^p) M_{p,s} f(x_0)^p$, and hence $\tau < M_{p,s} f(x_0)$.

COROLLARY 1. *Under the hypothesis of Theorem* 2, $M_{p,\infty} M_{r,s} f(x) = M_{r,s} f(x)$ *a.e.* $x \in \mathbf{R}^n$.

PROOF. At every point x of approximate continuity of a nonnegative measurable function ϕ, $M_{p,\infty}\phi(x) \geqslant \phi(x)$.

REMARKS. (i) The restriction $r \geqslant p$ in Theorem 2 is necessary. As an example, let $f(x) = x^{-2/3}\chi_{[0,1]}(x)$, and let $r = s = 1$, $p = 2$, $q = \infty$. For $0 < x < 1$, $Mf(x) \approx f(x)$, $M = M_{11}$, but the $L(2, \infty)$ norm of f is infinite.

(ii) The restriction $r \geqslant s$ is also necessary. An example for $p = r = 1$, $q = s = \infty$ can be found in [**1**].

4. We will now study the nondyadic version of the above results, and see that they are still valid with an additional constant depending upon n. However, one cannot expect an equality as in Corollary 1.

We retain the same notation as before and let $M_{p,q} f(x)$ be the above maximal function where the sup is now extended over all cubes Q with $x \in Q$.

THEOREM 3. *If* $1 \leqslant p < r < \infty$, $1 \leqslant s \leqslant r$, *and* $1 \leqslant q < \infty$, *then* $M_{p,q} M_{r,s} f(x) \leqslant c M_{r,s} f(x)$, $x \in \mathbf{R}^n$, *where c depends only upon* n, r, p, q.

PROOF. Fix $x_0 \in \mathbf{R}^n$, and let $\tau < M_{p,q}M_{r,s}f(x_0)$. As in the proof of Theorem 1, there is $x_0 \in Q$ such that

$$\tau^q \leqslant \frac{1}{|Q|^{q/p}}\left(\int_0^A + \int_A^\infty\right)\lambda_{(M_{r,s}f)\chi_Q}(y)^{q/p}\,d(y^q),$$

and again $(1/|Q|^{q/p})\int_0^A \leqslant A^q$.

We now use a familiar technique [3] and break up f as $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{\mathbf{R}^n \setminus 2Q}$. Since $M_{r,s}f(x) \leqslant 2(M_{r,s}f_1(x) + M_{r,s}f_2(x))$, we see that

$$\lambda_{M_{r,s}f}(y) \leqslant \lambda_{M_{r,s}}f_1\left(\frac{y}{4}\right) + \lambda_{M_{r,s}}f_2\left(\frac{y}{4}\right).$$

Hence the integral $\int_A^\infty$ is majorized by two integrals, one with f_1, the other with f_2.

For the integral with f_1, let $E_y = \{x\colon\ M_{r,s}f_1(x) > y\}$. Then there is a disjoint collection of cubes $\{Q_j\}$ such that $|E_y| \leqslant c_n \Sigma |Q_j|$ and

$$y < \left(1/|Q_j|^{1/r}\right)\left(\int_0^\infty \lambda_{f_1\chi_{Q_j}}(t)^{s/r}\,d(t^s)\right)^{1/s}.$$

As in the proof of Theorem 1,

$$|E_y| \leqslant c\frac{1}{y^r}\sum(\ )^{r/s} \leqslant c\frac{1}{y^r}\left\{\int_0^\infty \lambda_{f_1\chi \cup Q_j}(t)^{s/r}\,d(t^s)\right\}^{r/s}.$$

From this we get that

$$\begin{aligned}\frac{\lambda_{(M_{r,s}f_1)\chi_Q}(y)}{|Q|} &\leqslant \frac{c}{|Q|}\frac{1}{y^r}\left(\int_0^\infty \lambda_{f\chi_{2Q}}(y)^{s/r}\,d(t^s)\right)^{r/s}\\ &\leqslant \frac{c}{y^r}M_{r,s}f(x_0)^r.\end{aligned}$$

The integral $\int_A^\infty \lambda_{(M_{r,s}f_2)\chi_Q}(y/4)^{q/p}d(y^q)$ is easily disposed of. First we note that $M_{r,s}f_2$ is essentially constant on Q, i.e., $cM_{r,s}f_2(x_0) \geqslant M_{r,s}f_2(x)$, $x \in Q$, with $c \approx 5^{n/r}$. If $E_y = \{x\colon\ M_{r,s}f_2(x) > y\}$, then

$$|E_y \cap Q| \leqslant |\{x \in Q\colon\ cM_{r,s}f(x_0) \geqslant y\}| \leqslant c^r|Q|\left(\frac{M_{r,s}f(x_0)}{y}\right)^r.$$

Hence again

$$\frac{1}{|Q|}\lambda_{(M_{r,s}f_2)\chi_Q}(y) \leqslant \frac{c^r}{y^r}M_{r,s}f(x_0)^r.$$

The proof can now be completed as in Theorem 1.

COROLLARY 2. *If $1 < r < \infty$ and $1 \leqslant s \leqslant r$, then $M_{r,s}f \in A_1$.*

PROOF. This follows from Theorem 4 with $p = q = 1$.

COROLLARY 3. *With the same hypothesis as Theorem 3, $M_{p,q}M_{r,s}f \in A_1$.*

PROOF. If $M = M_{1,1}$, then $MM_{p,q}M_{r,s}f \leqslant C_1MM_{r,s}f \leqslant C_2M_{r,s}f \leqslant C_2M_{p,q}M_{r,s}f$.

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