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k-TO-1 FUNCTIONS ON ARCS FOR k EVEN

JO W. HEATH

ABSTRACT. For exactly k-to-1 functions from $[0,1]$ into $[0,1]$:

(1) at least one discontinuity is required (Harrold),

(2) if $k = 2$, then infinitely many discontinuities are needed, for any Hausdorff image space (Heath),

(3) if $k = 4$, or if k is odd, then there is such a function with only one discontinuity (Katsuura and Kellum),

and, it is shown here that

(4) if k is even and $k > 4$, then there is such a function with only two discontinuities, and no such function exists with fewer discontinuities.

I. Introduction. A function is *k-to-1* if each point inverse has exactly k elements, and it is *at most k-to-1* if each point inverse has at most k elements.

Over 45 years ago, O. G. Harrold, Jr. proved in **[2]** that there is no continuous k-to-1 map from $[0,1]$ into $[0,1]$. In a recent paper **[4]**, H. Katsuura and K. Kellum demonstrate that for each odd positive integer k, and for $k = 4$, there is a k-to-1 function from $[0,1]$ into $[0,1]$ with exactly one discontinuity. They ask what is the minimum number of discontinuities for k-to-1 functions from $[0,1]$ into $[0,1]$ with k even. In **[3]** the author showed that any 2-to-1 function from $[0,1]$ to a Hausdorff space requires infinitely many discontinuities. Thus only even numbers greater than 4 need be considered. Theorems 1 and 2 answer the question raised by Katsuura and Kellum for even integers at least 6:

THEOREM 1. *If f: $[0,1] \to [0,1]$ is a k-to-1 function and k is an even integer with $k > 4$, then f has at least two discontinuities.*

THEOREM 2. *If k is an even integer greater than 4 then there is a k-to-1 function from $[0,1]$ into $[0,1]$ with only two discontinuities.*

II. Proof of Theorem 1.

LEMMA 1. *Suppose f: $(0,1) \to (0,1)$ is a continuous map at most k-to-1, p is in $(0,1)$, and d is a positive number. Then there is a number x with $x < p$ and $|x - p| < d$ such that:*

1. *either $f((x,p)) \subseteq (f(x), f(p))$, or $f((x,p)) \subseteq (f(p), f(x))$, and*

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2. *if I is any subinterval of* $(0, 1)$ *between* $f(x)$ *and* $f(p)$, *then there is a subinterval J of I such that every horizontal line* $\{y = c\}$ *with c in J intersects the graph of f between x and p an odd number of times.*

PROOF. Since $f^{-1}(f(p))$ is finite, there is a positive number $d' < d$ such that no point within d' of p maps to $f(p)$ except p. Choose any number x' less than p so that $|x' - p| < d'$. The set $f^{-1}(f(x'))$ is finite so there is an x with $x' \leqslant x < p$ and $f(x) = f(x')$ such that no point of (x, p) maps to $f(x')$. Part 1 is true for this x. Note that the part 1 property implies that each point in $(0, 1)$ is either a crossing point, a local maximum, or a local minimum for the graph of f.

Now suppose part 2 is false and suppose $f(x) < f(p)$. Then there is an interval I in $(f(x), f(p))$ such that every subinterval contains a number c where the line $\{y = c\}$ intersects the graph of f between x and p an even number of times. Let c_1 be such a number in I and let n_1 be the even number of times $\{y = c_1\}$ intersects the graph between x and p. Since the graph goes continuously from $(x, f(x))$ below $\{y = c_1\}$ to $(p, f(p))$ above, an odd number of these intersection points must be crossings, say j of them. Of the others, i are local minima and m are local maxima. Since $m + i$ is odd, one is larger, say $m > i$. Part 1 is true for each point in $f^{-1}(c_1)$ so there is an interval $I_2 = (t, c_1)$ in I small enough that if c is in I_2 then $\{y = c\}$ intersects the graph between x and p at least $j + 2m$ times. From the negation of part 2 there is a number c_2 in I_2 such that the cardinality of this intersection is even, say n_2. Then

$$n_2 > j + 2m > j + m + i = n_1.$$

This process can be continued until $n_a > k$. Since the line $\{y = c_a\}$ cannot intersect the graph of an at most k-to-1 map more than k times this is a contradiction.

Note 1. Obviously the same properties hold to the right of p.

Note 2. Part 1 is a strengthening of a lemma found in Katsuura and Kellum [**4**].

Now, to prove Theorem 1, suppose that f: $[0, 1] \to [0, 1]$ is a k-to-1 function, with k an even integer greater than 4, and f has only one discontinuity, q.

CLAIM 1. Without loss of generality it can be assumed that $f([0, 1])$ is connected.

If $q = 0$, then, since $f((0, 1])$ is connected and $f(q) = f(x)$ for some x in $(0, 1]$, the image of f is connected. If $0 < q < 1$, the connected sets $f([0, q))$ and $f((q, 1])$ are disjoint, and $f(q) = f(x)$ for some x less than q, then f restricted to $[0, q]$ is also a k-to-1 function with one discontinuity (Harrold proved there is at least one discontinuity) from $[0, q]$ to $[0, 1]$.

CLAIM 2. Without loss of generality it can be assumed that $f([0, 1]) = (0, 1)$.

Since the image is connected it is an interval with or without endpoints. Suppose 0 is in the image of f. One of 0 or 1 is not q, say $1 \neq q$. Let $z = 1$ if $f(1) = 0$ and $z = 0$ if $f(1) \neq 0$. There are k points that map to 0; n of them are neither $0, 1$, nor q, with $n > 2$ since $k > 5$. Then $|f^{-1}(0)| = k \leqslant n + z + 2$, where the 2 allows the possibility that q and 0 are different and both map to 0. Each of the n points are local minima for f. Since f restricted to each of $(0, q)$ and $(q, 1)$ is continuous and at most k-to-1, Lemma 1 holds. From part 1 of the lemma used at points of $f^{-1}(0)$

there is a number e close to 0 so that the line $\{y = e\}$ intersects the graph of f at least $2n + z$ times. Since it intersects the graph exactly k times the following is true:

$$n + z + 2 \geqslant k \geqslant 2n + z,$$

from which it follows that $2 > n$, a contradiction. Therefore 0 is not in the image of f.

CLAIM 3. The discontinuity q is in $(0,1)$ and the limit of the images of any sequence converging to q from the left is 0 and from the right is 1 (or the other way around). It was proved in Katsuura and Kellum [**4**] that each limit exists and is either 0 or 1. If both limits were 1, say, then there would be an interval of numbers $(0, e)$ not mapped onto, contradicting the fact that the image of f is $(0,1)$. For the same reason, since both 0 and 1 limits must be achieved, q must be interior to $[0,1]$ to have two sides in the domain.

CLAIM 4. $f(0) = f(1)$. If not, one of them, say $f(1)$, is not equal to $f(q)$. From Lemma 1 there are disjoint intervals (a_i, b_i), $i = 1, 2, \ldots, k-1$, about the points of $f^{-1}(f(1))$ other than 1 and an interval $(a_k, 1]$ disjoint from the others, that satisfy part 1.

Since points near q map to values near 0 or 1, there is a positive number e so that the e-neighborhood about the line $\{y = f(1)\}$ contains no point of the graph of f whose first coordinate lies outside the intervals (a_i, b_i) and $(a_k, 1)$.

From part 2 of the lemma there are numbers c in $(f(1), f(1) + e)$ and c' in $(f(1) - e, f(1))$ so that if x_i is the point of $f^{-1}(f(1))$ in (a_i, b_i) then

(1) if x_i is a crossing point for the graph of f on the line $\{y = f(1)\}$, then both of the lines $\{y = c\}$ and $\{y = c'\}$ intersect the graph of f an odd number of times between a_i and b_i,

(2) if x_i is a local minimum (or maximum) for f then the line $\{y = c\}$ (or $\{y = c'\}$) intersects the graph of f an even number of times between a_i and b_i and the other line $\{y = c'\}$ (or $\{y = c\}$) does not intersect the graph between a_i and b_i, and

(3) one of the lines $\{y = c\}$ and $\{y = c'\}$ intersects the graph an odd number of times between a_k and 1 and the other not at all. It will be assumed here that $\{y = c\}$ is the one that intersects the graph over $(a_k, 1]$.

Let m, n, and j denote the number of local maxima, local minima, and crossing points in $f^{-1}(f(1)) - \{1\}$, respectively. Then

$|f^{-1}(c)| = k$ is the sum of n even numbers plus $j + 1$ odd numbers, and

$|f^{-1}(c)| = k$ is the sum of m even numbers and j odd numbers.

Since k cannot be both even and odd there is a contradiction.

CLAIM 5. If Claim 3 is as stated and not the other way around, then $f([0, q)) \subseteq (0, f(0)]$, and $f((q, 1]) \subseteq [f(1), 1)$.

Suppose there is a point (a, b) on the graph of f with $0 < a < q$, $f(0) < b$, and b is the largest such value. The inverse of b contains m, say, local maxima points, and the number of crossing points, j, must be odd since they are all greater than q and the graph of f from q to 1 goes from y-coordinates arbitrarily close to 1 near q down to $f(1)$ at 1, and $f(1) < b$. As in Claim 4, there is a number $c' < b$, $c' \neq f(q)$, close enough to b that $|f^{-1}(c')| = k$ is the sum of m even numbers and j odd numbers, which is impossible for the even number k.

CLAIM 6. There is a contradiction.

With the graph of f to the left of q below and on $\{y = f(0)\}$ and the graph to the right of q above and on the same line, all of the points (m of them) of $f^{-1}(f(0))$ in $(0, q)$ are local maxima. Again, choose c' near $f(0)$ with $c' < f(0)$, $c' \neq f(q)$, and the line $\{y = c'\}$ intersects the graph in k points, the sum of an odd number, corresponding to the point $(0, f(0))$, plus m even numbers; a contradiction.

III. Proof of Theorem 2. The construction here is a generalization of an example in **[4]** which in turn used examples from **[1]** and **[2]**. The basic pieces of the graph of the function to be constructed are generalized W's and M's defined for a given integer n:

(i) Choose $2n+1$ distinct points in $[0,1]$, $0 = a_0 < a_1 < \cdots < a_{2n} = b$, and define $W(a_{2t}) = 1$ and $W(a_{2t+1}) = 0$ for relevant t. Denote by $W(n)$ the piecewise linear extension to $[0,1]$.

(ii) Given an interval $[a, b]$, choose $2n+2$ points $a = a_0 < a_1 < \cdots < a_{2n+1} = b$, and define $M(a_{2t}) = a$ and $M(a_{2t+1}) = b$ for relevant t. Let $M(a, b, n)$ denote the piecewise linear extension to a function from $[a, b]$ onto $[a, b]$. These basic pieces will be used to describe the basic units $B(n)$ and $A(t, n)$ for integers $t < n$:

(iii) Define the function $B_1(n)$ from $[0,1]$ to $[0,1]$ by

$$B_1(n) = (0,0) + \bigcup_{i=0}^{\infty} M(2^{-(i+1)}, 2^{-i}, n).$$

Get the graph of $B_2(n)$ by first reflecting the graph of $B_1(n)$ about the vertical line $\{x = .5\}$ and then about the horizontal line $\{y = .5\}$.

(iv) Divide the rectangle $[0,3] \times [0,2]$ into six unit squares using matrix $S(i, j)$ notation. In the lower left square, $S(2,1)$, put the graph of $B_1(m)$; in $S(2,2)$ put the graph of $W(t)$; and in the upper right square, $S(1,3)$, put the graph of $B_2(n)$.

Consider the cardinality, $I(c)$, of the intersection of the line $\{y = c\}$ with this composite graph: If $1 < c < 2$ then $I(c) = 2n+1$. $I(1) = m + t + n + 1$. If $0 < c < 1$ then $I(c) = 2m + 1 + 2t$.

For the interior intersections to be constant, $2n + 1 = m + t + n + 1 = 2m + 1 + 2t$ is needed, i.e. $m + t = n$.

Given n and t, then, set $m = n - t$ and denote the described function on $[0,3]$ by $A_1(t, n)$. Denote by $A_2(t, n)$ the function constructed by reflecting the graph of $A_1(t,n)$ about the line $\{x = 1.5\}$, and by $A_3(t, n)$ the function constructed by reflecting the graph of $A_1(t, n)$ about $\{y = 1\}$.

Finally, since k is an even integer greater than 4 it can be written as $(2r + 1) + (2s + 1)$ where r and s are both positive. Define a continuous function g from $[0,12]$ to $[0,4]$ by dividing $[0,12] \times [0,4]$ into eight 2×3 rectangles. Put $A_1(r-1, r)$ in the upper left rectangle $T(1,1)$, $A_2(s-1, s)$ in $T(1,2)$, $A_3(r-1, r)$ in $T(2,3)$, and $B_1(s)$ in the lower right rectangle $T(2,4)$.

The wanted function f will equal g everywhere except at its two discontinuities, $f(3) = 2$ and $f(9) = 2$. The careful reader will agree that every point inverse has k elements.

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DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36849