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THE PREPARATION THEOREM FOR COMPOSITE FUNCTIONS

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ABSTRACT. We present a simple extension of the preparation theorem of B. Malgrange and J. Mather to the case of composite functions. As a corollary we obtain a short proof of the equivariant preparation theorem of V. Poénaru.

1. Formulation of the results. We denote by $\mathcal{E}(p)$ the ring of germs of C^∞-functions at $0 \in \mathbf{R}^p$, by $\mathcal{E}(p,q)$ the set of germs at $0 \in \mathbf{R}^p$ of C^∞-transformations $\mathbf{R}^p \longrightarrow \mathbf{R}^q$ which preserve the origin, and by $\mathbf{m}(p)$ the maximal ideal of $\mathcal{E}(p)$ formed by all functions vanishing at 0. A transformation $H \in \mathcal{E}(p,q)$ induces a local ring homomorphism $\mathcal{E}(q) \xrightarrow{H^*} \mathcal{E}(p)$ defined by $\beta \mapsto \beta \circ H$.

For $\rho \in \mathcal{E}(m,k)$ and $\eta \in \mathcal{E}(n,l)$ we introduce $\mathcal{E}_\rho(m) \stackrel{\text{def}}{=} \{\alpha \circ \rho; \alpha \in \mathcal{E}(k)\}$, $\mathcal{E}_\eta(n) \stackrel{\text{def}}{=} \{\beta \circ \eta; \beta \in \mathcal{E}(l)\}$, $\mathbf{m}_\rho(m) \stackrel{\text{def}}{=} \mathbf{m}(m) \cap \mathcal{E}_\rho(m)$, and $\mathbf{m}_\eta(n) \stackrel{\text{def}}{=} \mathbf{m}(n) \cap \mathcal{E}_\eta(n)$. Obviously $\mathbf{m}_\eta(n) = \eta^*\mathbf{m}(l)$ and $\mathbf{m}_\rho(m) = \rho^*\mathbf{m}(k)$.

A germ $f \in \mathcal{E}(m,n)$ such that $f^*\mathcal{E}_\eta(n) \subset \mathcal{E}_\rho(m)$ will be called a $\rho\eta$-germ; for such a transformation each component of $\eta \circ f$ belongs to $\mathcal{E}_\rho(m)$ and so is of the form $F_i \circ \rho$. Hence there exists $F \in \mathcal{E}(k,l)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{R}^k & \xrightarrow{F} & \mathbf{R}^l \\ \uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \eta} \\ \mathbf{R}^m & \xrightarrow{f} & \mathbf{R}^n \end{array} \tag{1}$$

As $\rho^*\mathcal{E}(k)$ and $f^*\mathcal{E}_\eta(n) = (\eta \circ f)^*\mathcal{E}(l) = (F \circ \rho)^*\mathcal{E}(l)$ are all subrings of $\mathcal{E}_\rho(m)$, any $\mathcal{E}_\rho(m)$-module A could be considered as a $\mathcal{E}(k)$-, $\mathcal{E}_\rho(n)$- or $\mathcal{E}(l)$-module. Obviously $a_1, \ldots, a_p$ generate A as an $\mathcal{E}_\rho(m)$-module (respectively an $\mathcal{E}_\eta(n)$-module) if and only if they generate it as an $\mathcal{E}(k)$-module ($\mathcal{E}(l)$-module, respectively). A similar remark concerns the generators of the isomorphic vector spaces

$$A/F^*\mathbf{m}(l) \cdot A \approx A/(\eta \circ f)^*\mathbf{m}(l) \cdot A \approx A/f^*\mathbf{m}_\eta(n) \cdot A.$$

From the preparation theorem [**1**, p. 59, **2**, **3**] applied to F we obtain the following result.

THEOREM 1. *Let $\rho \in \mathcal{E}(m,k)$, $\eta \in \mathcal{E}(n,l)$, let A be a finitely generated $\mathcal{E}_\rho(m)$-module, let $f \in \mathcal{E}(m,n)$ be a $\rho\eta$-germ, and suppose $F \in \mathcal{E}(k,l)$ makes the diagram*

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(1) *commute. Then the elements* $a_1, \dots, a_p$ *generate* A *as an* $\mathcal{E}_\eta(n)$*-module if and only if they represent generators of the real vector space* $A/(f^*\mathbf{m}_\eta(n) \cdot A)$.

Let $\mathbf{m}(\rho, k)$ denote the ideal in $\eta(k)$ of all functions vanishing on $\rho(\mathbf{R}^m)$. Note that the ring $\mathcal{E}_\rho(m)$ is isomorphic to $\mathcal{E}(k)/\mathbf{m}(\rho, k)$. Hence for any $\mathcal{E}(k)$-module A^* the factor module $A^*/(\mathbf{m}(\rho, k) \cdot A^*)$ has the natural structure of an $\mathcal{E}_\rho(m)$-module.

COROLLARY 1. *With the hypotheses of Theorem* 1 *let* $A \stackrel{\text{def}}{=} A^*/\mathbf{m}(\rho, k) \cdot A^*$, *where* A^* *is a finitely generated* $\mathcal{E}(k)$*-module. Then* $a_1, \dots, a_p$ *belonging to* A^* *represent generators of the* $\mathcal{E}_\eta(n)$*-module* A *if and only if they represent generators of the real vector space*

$$A^*/(\mathbf{m}(\rho, k) \cdot A^* + F^*\mathbf{m}(l) \cdot A^*).$$

PROOF. Let us denote by A_1 the $\mathcal{E}(k)$-submodule $\mathbf{m}(\rho, k) \cdot A^*$ of A^*. From the following sequence of the natural $\mathcal{E}(k)$-module isomorphisms

$$\begin{aligned} A/f^*\mathbf{m}_\eta(n) \cdot A &= A/\rho^*F^*\mathbf{m}(l) \cdot A \approx A/F^*\mathbf{m}(l) \cdot A \\ &\approx (A^*/A_1)/((F^*\mathbf{m}(l) \cdot A^*)/A_1) \approx A^*/(F^*\mathbf{m}(l) \cdot A^* + A_1), \end{aligned}$$

it follows that the vector spaces $A/f^*\mathbf{m}_\eta(n) \cdot A$ and $A^*/(F^*\mathbf{m}(l) \cdot A^* + \mathbf{m}(\rho, k) \cdot A^*)$ are isomorphic. Now we can refer to Theorem 1, since A is a finitely generated $\mathcal{E}_\rho(m)$-module (because A^* is finitely generated over $\mathcal{E}(k)$).

2. Equivariant division theorem. This paragraph provides some examples of applications of Theorem 1.

Consider a compact Lie group G acting orthogonally on $\mathbf{R}^m$ and $\mathbf{R}^n$. According to G. Schwarz [**5**] there exist polynomial maps $\rho \in \mathcal{E}(m, k)$ and $\eta \in \mathcal{E}(n, l)$, called Hilbert maps, such that $\mathcal{E}_\rho(m)$ and $\mathcal{E}_\eta(n)$ are exactly the sets of G-invariant germs $\mathcal{E}_G(m)$ and $\mathcal{E}_G(n)$, respectively. Denote $\mathbf{m}_G(n) \stackrel{\text{def}}{=} \mathbf{m}(n) \cap \mathcal{E}_G(n)$. Obviously any G-equivariant $f \in \mathcal{E}(m, n)$ is a $\rho\eta$-germ, so from Theorem 1 there follows the equivariant preparation theorem [**4**].

THEOREM 2. *If* A *is a finitely generated* $\mathcal{E}_G(m)$*-module then* A *is finitely generated as a* $\mathcal{E}_G(n)$*-module if and only if the real vector space* $A/f^*\mathbf{m}_G(n)A$ *has a finite dimension.*

EXAMPLE. Let $\mathbf{R}^m = \mathbf{R}^n = \mathbf{R}^2$. Let $G = Z_2 = \{\pm 1\}$ operate on $\mathbf{R}^2$ as $(x, y) \mapsto (\varepsilon x, \varepsilon y)$ for $\varepsilon \in G$. Let us consider an equivariant transformation f: $\mathbf{R}^2 \to \mathbf{R}^2$, $(x, y) \mapsto (x, x^3 + y^3)$. Using Corollary 1 we shall show that $1, xy, y^2, y^4$ generate $\mathcal{E}_{Z_2}(2)$ over $f^*\mathcal{E}_{Z_2}(2)$.

The Hilbert maps $\rho = \eta\colon \mathbf{R}^2 \to \mathbf{R}^3$ could be defined as $(x, y) \mapsto (x^2, y^2, xy)$. The set $\rho(\mathbf{R}^2) \subset \mathbf{R}^3$ is a semicone $uv = z^2$, $u \geq 0$, $v \geq 0$, where u, v, z are coordinates in $\mathbf{R}^3$. Obviously $\mathcal{E}_{Z_2}(2) = \rho^*\mathcal{E}(3) \approx \mathcal{E}(3)/\mathbf{m}(\rho, 3)$. Transformation f is a $\rho\rho$-germ and $\rho \circ f(x, y) = (x^2, (x^3 + y^3)^2, x^4 + xy^3)$ is a Z_2-invariant mapping. A mapping F: $\mathbf{R}^3 \to \mathbf{R}^3$ as in (1), i.e. such that $\rho \circ f = F \circ \rho$, could be defined by $(u, v, z) \mapsto (u, u^3 + 2z^3 + v^3, u^2 + zv)$. Let us consider the ideal $I \stackrel{\text{def}}{=} \langle u, z^2, zv, v^3 \rangle_{\mathcal{E}(3)}$. By straightforward checking we get

$$I = \langle u, u^3 + 2z^3 + v^3, u^2 + zv, uv - z^2 \rangle_{\mathcal{E}(3)} \subset (F^*\mathbf{m}(3) \cdot \mathcal{E}(3) + \mathbf{m}(\rho, 3) \cdot \mathcal{E}(3)).$$

It is easy to observe that $\mathbf{m}^3(3) \subset I + \mathbf{m}^4(3)$, so $\mathbf{m}^3(3) \subset I$ by Nakayama's lemma [**1**]. Now a simple computation shows that $1, z, v, v^2$ represent generators of the real vector space $\mathcal{E}(3)/I$ and so they generate $\mathcal{E}(3)/(F^*\mathbf{m}(3) \cdot \mathcal{E}(3) + \mathbf{m}(\rho, 3))$, the real vector space. By Corollary 1 (for $A^* = \mathcal{E}(3)$ and $A = \mathcal{E}(3)/\mathbf{m}(\rho, 3))$ they represent generators of module A over $f^*\mathcal{E}_{Z_2}(2) \approx \mathcal{E}_\eta(2)$. Now considering an $f^*\mathcal{E}_{Z_2}(2)$-module isomorphism $\rho^*\colon A \to \mathcal{E}_{Z_2}(2)$ we find that their combinations with ρ, i.e. $1, xy, y^2, y^4$ generate $\mathcal{E}_{Z_2}(2)$ over $f^*\mathcal{E}_{Z_2}(2)$.

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ADDED IN PROOF. I learned recently that a similar derivation of equivariant division theorem has been published by J. Damon (*The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$*, Proc. Sympos. Pure Math., vol. 40, Part 1, Amer. Math. Soc., Providence, R.I., 1983, pp. 233–254).

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