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ON THE AREA OF THE REGION WHERE AN ENTIRE FUNCTION IS GREATER THAN ONE

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To the memory of Professor Robert L. Long

ABSTRACT. Using Carleman's inequality, we prove that if f is entire and of finite order $\rho \geq 1$, then

$$\limsup_{r\to\infty} \frac{A(r)}{r^2} \geq \frac{\pi}{2\rho},$$

where $A(r)$ is the area of the region $\{z: |f(z)| \geq 1 \text{ and } |z| \leq r\}$.

1. Introduction. In [**2**], Edrei and Erdös proved the following

THEOREM A. *Let f be an entire function and $D = \{z: |f(z)| > B\}$ $(B > 0)$. If there exists a positive number B such that the area of D is finite, then*

$$\liminf_{r\to\infty} \frac{\ln\ln\ln M(r,f)}{\ln r} \geq 2. \tag{1.1}$$

In this brief note, we will establish the following

THEOREM 1. *Let f be an entire function of order ρ, $1 \leq \rho < \infty$, and let $A(r)$ denote the area of the region*

$$D_r = \{z: |f(z)| \geq 1, |z| \leq r\}.$$

Then

$$\limsup_{r\to\infty} \frac{A(r)}{r^2} \geq \frac{\pi}{2\rho}.$$

The method, which is different from that of Edrei and Erdös, is based on Carleman's famous inequality which we are about to introduce.

Let D be a region on the complex plane. The boundary of D consists of a finite or infinite number of analytic curves clustering nowhere in the finite complex plane. For any r, $0 < r < \infty$, we denote by D_r the part of D lying in $|z| < r$. Let $A_k(r)$ $(k = 1, 2, \ldots, n(r))$ be the arcs of $|z| = r$ contained in D and $r\theta_k(r)$ be their arc lengths. Let $E = \{r: |z| = r$ is contained wholly in $D\}$ and $E^c = [0,\infty) - E$. If $r \in E^c$, we define

$$\theta(r) = \max_k \theta_k(r).$$

For the moment, we leave $\theta(r)$ undefined if $r \in E$.

We now state the following version of Carleman's inequality due to K. Arima [**1**, p. 64].

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THEOREM B. *Let f be an entire function and D be the region where $|f(z)| > 1$. Let $\theta(r)$ be defined as before for the region D. Then for any a, $0 < a < 1$, we have*

$$\ln\ln M(r,f) > \pi \int_{E_r^c} \frac{dr}{r\theta(r)} - c, \tag{1.2}$$

where $E_r^c = E^c \cap [1, ar]$ and the constant c depends on a only.

2. New proof of Theorem A. Without loss of generality, we assume $B = 1$. We first choose an a, $0 < a < 1$ (a will be fixed throughout §§2 and 3). Let $E_r = E \cap [1, ar]$, and define $\theta(r) = 2\pi$ for $r \in E$. Then

$$\pi \int_{E_r} \frac{dt}{t\theta(t)} = \frac{1}{2} \int_{E_r} \frac{dt}{t} \leq \frac{1}{2} \ln ar. \tag{2.1}$$

From (1.2) and (2.1), we have

$$\pi \int_1^{ar} \frac{dt}{t\theta(t)} < \ln\ln M(r,f) + \frac{1}{2} \ln ar + c. \tag{2.2}$$

By Schwarz's inequality,

$$(ar-1)^2 = \left(\int_1^{ar} dt\right)^2 \leq \int_1^{ar} t\theta(t)\, dt \cdot \int_1^{ar} \frac{dt}{t\theta(t)}. \tag{2.3}$$

We recall that $A(r) =$ area of $\{z \colon |f(z)| \geq 1 \text{ and } |z| \leq r\}$. Clearly

$$\int_1^{ar} t\theta(t)\, dt \leq A(ar). \tag{2.4}$$

From (2.2), (2.3) and (2.4), we obtain

$$(ar-1)^2 < (A(ar)/\pi) \left(\ln\ln M(r,f) + \tfrac{1}{2} \ln ar + c\right). \tag{2.5}$$

If the area of D is finite, (2.5) clearly implies (1.1). This completes the proof of Theorem A.

3. Proof of Theorem 1. Let

$$\mu = \liminf_{r\to\infty} \frac{1}{\ln r} \int_{E_r} \frac{dt}{t}.$$

Then $0 \leq \mu \leq 1$.

We will prove the following proposition which is slightly more general than Theorem 1.

PROPOSITION 1. *Let f be an entire function of order $\rho > 0$. Then*

$$\limsup_{r\to\infty} \frac{A(r)}{r^2} \geq \pi\mu + \frac{\pi(1-\mu)^2}{2\rho}.$$

PROOF. From Schwarz's inequality,

$$\left(\int_{E_r^c} \frac{1}{t}\, dt\right)^2 \leq \int_{E_r^c} \frac{\theta(t)}{t}\, dt \cdot \int_{E_r^c} \frac{1}{t\theta(t)}\, dt. \tag{3.1}$$

Combining (3.1) with Carleman's inequality, we obtain

$$\int_{E_r^c} \frac{\theta(t)}{t}\, dt \geq \pi \left(\int_{E_r^c} \frac{dt}{t}\right)^2 \Big/ (\ln\ln M(r,f) + c). \tag{3.2}$$

We again define $\theta(r) = 2\pi$ for $r \in E$. Then

$$\begin{aligned}\int_{E_r^c} \frac{\theta(t)}{t}\,dt &= \int_1^{ar} \frac{\theta(t)}{t}\,dt - \int_{E_r} \frac{\theta(t)}{t}\,dt \\ &= \int_1^{ar} \frac{\theta(t)}{t}\,dt - 2\pi \int_{E_r} \frac{dt}{t}.\end{aligned} \tag{3.3}$$

Let $B(r) = \int_1^r t\theta(t)\,dt$. Clearly, $B(r) \le A(r)$ for all $r \ge 1$. We therefore have

$$\begin{aligned}\int_1^{ar} \frac{\theta(t)}{t}\,dt &= \int_1^{ar} \frac{dB(t)}{t^2} = 2\int_1^{ar} \frac{B(t)}{t^3}\,dt + K(r) \\ &\le 2\int_1^{ar} \frac{A(t)}{t^3}\,dt + K(r),\end{aligned} \tag{3.4}$$

where $K(r) = (B(ar)/a^2r^2 - B(1))$. We note that $B(r)/r^2 \le \pi$ for all r. From (3.2), (3.3) and (3.4), we conclude that
(3.5)

$$\frac{K(r)}{\ln r} + \frac{2}{\ln r}\int_1^{ar} \frac{A(t)}{t^3}\,dt \ge \frac{2\pi}{\ln r}\int_{E_r} \frac{dt}{t} + \pi\left(\frac{1}{\ln r}\int_{E_r^c} \frac{dt}{t}\right)^2 \Bigg/ \left(\frac{\ln\ln M(r,f) + c}{\ln r}\right).$$

It follows immediately from (3.5) that

$$\limsup_{r\to\infty} \frac{A(r)}{r^2} \ge \pi\mu + \frac{\pi(1-\mu)^2}{2\rho}.$$

This finishes the proof of Proposition 1.

It is easy to verify that, if $\rho \ge 1$.

$$\pi\mu + \frac{\pi(1-\mu)^2}{2\rho} \ge \frac{\pi}{2\rho}. \tag{3.6}$$

Theorem 1 follows from (3.6).

REMARK. From (3.5), we see that if $\rho = 0$, then $\mu = 1$. This yields

$$\limsup_{r\to\infty} \frac{A(r)}{r^2} = \pi.$$

We also note that Proposition 1 gives $\mu + (1-\mu)^2/2\rho \le 1$ for $0 < \rho < 1$. This provides the following relation for μ and ρ:

$$1 - \frac{\rho}{2} \le \mu \le 1.$$

We also point out here that the conclusion of Theorem 1 is sharp, as may be seen by considering Mittag-Leffler's function $E_{1/\rho}$.

REFERENCES

1. K. Arima, *On maximum modulus of integral functions*, J. Math. Soc. Japan **4** (1952), 62–66.
2. A. Edrei and P. Erdös, *Entire functions bounded outside a finite area*, Acta Math. Hungar. **45** (1985), 367–376.

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