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COMPLEMENTED INVARIANT SUBSPACES OF H^p, $0 < p < 1$, AND THE HAHN-BANACH EXTENSION PROPERTY

WILLIAM S. COHN

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ABSTRACT. Let $0 < p < 1$ and let H^p denote the usual Hardy class of functions analytic on the disc. In this note we show that an invariant subspace of H^p is complemented in H^p if and only if it has the form BH^p where B is a Blaschke product whose zero sequence is a Carleson sequence. We also prove that this occurs if and only if the invariant subspace has the Hahn-Banach extension property.

If $\mathcal{A}$ is the disc algebra of functions analytic in the open unit disc and continuous on the closed disc then the following theorem characterizes those closed ideals in $\mathcal{A}$ which are also complemented in $\mathcal{A}$.

THEOREM A (CASAZZA, PENGRA, SUNDBERG [**1**]). *For I a closed ideal in $\mathcal{A}$, there exists a bounded projection Q of $\mathcal{A}$ onto I if and only if there is a Blaschke product B whose zero sequence is a Carleson sequence and $I = \{f : f \in \mathcal{A}$ and B divides $f\}$.*

Recall that a sequence $\{z_n\}$ of points in the disc is a Carleson sequence if

$$\mu = \sum_{n=1}^{\infty} (1 - |z_n|)\delta_{z_n}$$

is a Carleson measure (see [**3**, Chapter VI]).

The authors in [**1**] remark that the complemented weak* closed invariant subspaces of H^∞ have a similar characterization; they are of the form BH^∞ where B is a Blaschke product as in Theorem A.

If $1 < p < \infty$, the M. Riesz theorem shows that for any inner function ϕ, ϕH^p is complemented in H^p. A bounded projection is given by

$$Q(f) = \phi P(\bar{\phi} f)$$

where P is the Riesz projection of L^p onto H^p.

In this note, we show that for $0 < p \leq 1$ the complemented invariant subspaces of H^p are like the ones in H^∞.

THEOREM B. *Let $0 < p \leq 1$. If ϕ is an inner function then there exists a bounded projection of H^p onto ϕH^p if and only if $\phi = B$, where B is a Blaschke product whose zero sequence forms a Carleson sequence.*

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PROOF. A careful examination of the argument given in [**1**] to prove necessity in Theorem A reveals that it can be applied (with minor modifications) to prove necessity in Theorem B for $p = 1$. For $0 < p < 1$ this can also be done, although the modifications necessary are slightly more difficult; we offer a different approach which depends on some other ideas in the literature and also provides more information.

For $0 < p < 1$, H^p, while not locally convex, is a locally bounded space whose dual separates points (see [**5**, p. 6] for the definition of locally bounded). The results of Duren, Romberg, and Shields in [**2**] show that the closure of the canonical imbedding of H^p in the Banach space $(H^p)^{**}$ is the space B_p of functions f analytic in the disc such that

$$||f||_{B_p} = \int_0^{2\pi} \int_0^1 |f(re^{i\theta})|(1-r)^{-2+1/p}\, dr\, d\theta < \infty.$$

Suppose that ϕH^p is complemented in H^p where $0 < p < 1$. Then it is clear that ϕH^p has the Hahn-Banach extension property (HBEP) in H^p, that is, if λ is a continuous linear functional on ϕH^p, then λ is the restriction to ϕH^p of a continuous linear functional Λ on H^p. Let $T\colon H^p \to \phi H^p$ be defined by $T(f) = \phi f$. Then T is an isomorphism of H^p onto ϕH^p. We claim that it follows now that the operator $S\colon B_p \to \phi B_p$ defined by $S(f) = \phi f$ is bounded below, i.e. there is a constant $c > 0$ such that

$$(*) \qquad ||\phi f||_{B_p} \geq c||f||_{B_p}.$$

It is a result due essentially to Horowitz [**4**, Theorem 2] (see also [**6**]) that $(*)$ holds if and only if ϕ is a Blaschke product whose zero sequence is Carleson. If we observe that $S = T^{**} = (T^*)^*$ restricted to B_p where * denotes the adjoint operator, then necessity in Theorem B for $0 < p < 1$ follows from the following result.

THEOREM C. *Let $(X, ||\ ||)$ be a locally bounded space with quasi-norm $||\ ||$, whose dual separates points. Suppose $T\colon X \to Y$ is an isomorphism of X onto a closed subspace $Y \subseteq X$. Then the following properties are equivalent.*

(1) *Y has the HBEP as a subspace of X.*

(2) *$T^{**}\colon X \mapsto Y$ is bounded below, i.e. there exists a constant $c > 0$ such that for all $x \in X$,*

$$||Tx||_{X^{**}} \geq c||x||_{X^{**}}.$$

Here $||\ ||_{X^{**}}$ denotes the usual norm on X^{**}, and if $x \in X$, by $||x||_{X^{**}}$, we mean the X^{**} norm of the image of x in X^{**} under the canonical imbedding. A similar remark holds for $||y||_{Y^{**}}$ if $y \in Y$.

PROOF. Suppose (2) holds. If $x \in X$ then $T^{**}x = Tx$. Thus

$$||Tx||_{Y^{**}} \leq ||T^{**}||\ ||x||_{X^{**}} \leq c^{-1}||T^{**}||\ ||Tx||_{X^{**}}.$$

Since $T(X) = Y$, it follows that for a $c_1 > 0$,

$$||y||_{Y^{**}} \leq c_1||y||_{X^{**}}$$

for all $y \in Y$.

The reverse inequality

$$||y||_{X^{**}} \le c_2||y||_{Y^{**}}$$

follows from the boundedness of $i^{**}: Y^{**} \to X^{**}$ where $i: Y \to X$ denotes inclusion. Thus the $||\ ||_{X^{**}}$ and $||\ ||_{Y^{**}}$ topologies agree on Y. If $\varphi \in Y^*$ then φ belongs to $(Y, ||\ ||_{Y^{**}})^*$ (the dual of Y with the $||\ ||_{Y^{**}}$-normed topology; see [**5**, pp. 27–28]). Thus $\varphi \in Y^*$ implies that $\varphi \in (Y, ||\ ||_{X^{**}})^*$. Since $(Y, ||\ ||_{X^{**}})$ is a topological subspace of the normed space $(X, ||\ ||_{X^{**}})$ the Hahn-Banach theorem implies that φ is the restriction to Y of some $\Phi \in (X, ||\ ||_{X^{**}})^* = X^*$. Thus (2) implies (1).

Conversely, suppose (1) holds. Letting $i: Y \to X$ again denote inclusion, (1) implies that $i^*: X^* \to Y^*$ is onto and therefore $i^{**}: Y^{**} \to X^{**}$ is bounded below. Thus

$$||y||_{X^{**}} \ge c||y||_{Y^{**}}$$

for $c > 0$ and $y \in Y$. Use the fact that T is an isomorphism to get that

$$||Tx||_{Y^{**}} \ge c_1||x||_{X^{**}}$$

for some $c_1 > 0$ and set $y = Tx$ to conclude that

$$||Tx||_{X^{**}} \ge cc_1||x||_{X^{**}}.$$

Thus (1) implies (2) and the proof of Theorem C is complete.

It remains to prove sufficiency in Theorem B. If ϕ is a Blaschke product whose zero sequence is a Carleson sequence, then by [**6**, Lemma 21] we may factor ϕ as $\phi = \phi_1\phi_2\cdots\phi_m$ where the zero sequence of each ϕ_i is an interpolation sequence. Suppose $\{z_n\}$ is the zero sequence of ϕ_i. Define, for $0 < p \le 1$,

$$(Q_i f)(z) = f(z) - \sum \frac{b_n(z)f(z_n)}{b_n(z_n)} \frac{(1-|z_n|^2)^{2/p}}{(1-\bar{z}_n z)^{2/p}},$$

where $b_n(z) = \prod_{k\ne n}(\bar{z}_k/|z_k|)(z_k - z)/(1-\bar{z}_k z)$.

An application of the triangle inequality and the fact that

$$\sum |f(z_n)|^p(1-|z_n|) \le c||f||_p^p$$

for an absolute constant c, shows that $Q_i: H^p \to H^p$ is bounded. It is a simple matter to prove that Q_i is also a projection of H^p onto $\phi_i H^p$. Now for $k = 2, \dots, m$ define

$$L_k f = \phi_1\phi_2\cdots\phi_{k-1}Q_k(\bar{\phi}_1\bar{\phi}_2\cdots\bar{\phi}_{k-1}f).$$

It follows that

$$Q = L_m L_{m-1}\cdots L_2 Q_1$$

is bounded projection of H^p onto ϕH^p. This completes the proof of Theorem B. The following curious fact has been established as well.

COROLLARY. *If $0 < p < 1$, then an invariant subspace of H^p is complemented in H^p if and only if it has the HBEP as a subspace of H^p.*

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DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202