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# ON CLASSIFICATION OF QUADRATIC HARMONIC MAPS OF $S^3$

GABOR TOTH

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ABSTRACT. By the generalized Do Carmo-Wallach classification theorem polynomial harmonic maps between spheres can be parametrized by a finite-dimensional compact convex body. Here we describe the boundary of the parameter space in the first nonrigid range by exhibiting a large number of quadratic harmonic maps of $S^3$ into spheres.

**1. Statement of the result.** A fundamental problem in harmonic map theory is to classify all harmonic maps $f\colon S^m \to S^n$ between Euclidean spheres whose components are homogeneous harmonic polynomials of (fixed) degree $k$ (cf. [**2**, **4**, **5**] and [**3**, Problem (4.4), p. 70]. By the generalized Do Carmo-Wallach classification theorem, for fixed $m$ and $k$, the equivalence classes of full harmonic polynomial maps of degree $k$ can be parametrized by a compact convex body $L^0$ lying in a finite-dimensional vector space $E$ [**6**, pp. 297–304]. Moreover, $\dim E = 0$ iff $k = 1$ and $m \geq 2$ (rigidity of isometries) or $m = 2$ and $k \geq 1$ (Calabi's rigidity theorem [**1**, **7**]). In the nonrigid range $m > 2$ and $k > 1$, though the decomposition of the $SO(m+1)$-module structure of $E \otimes_{\mathbf{R}} \mathbf{C}$ (induced from the $SO(m+1)$-action on $L^0$ by precomposing harmonic maps with isometries of $S^m$) into irreducible components is known [**6**], the orbit structure of the invariant subspace $L^0$ (especially that of $\partial L^0$) is rather subtle. It is then natural to consider the lowest-dimensional case $m = 3$ and $k = 2$ ($\dim E = 10$), i.e., to study full quadratic harmonic maps $f\colon S^3 \to S^n$, $2 \leq n \leq 8$.

THEOREM. (i) *Any full quadratic harmonic map $f\colon S^3 \to S^2$ is globally rigid, i.e., there exist $U \in O(4)$ and $V \in O(3)$ such that $V \circ f \circ U$ is the Hopf map*;

(ii) *There is no full quadratic harmonic map $f\colon S^3 \to S^3$*;

(iii) *For $4 \leq n \leq 8$, there exist nonglobally rigid full quadratic harmonic maps $f_n\colon S^3 \to S^n$.*

REMARK. By way of contrast (to (ii)), for the existence of polynomial (nonharmonic) maps $f\colon S^3 \to S^3$, see [**8**].

**2. Proof.** The entire space of quadratic harmonic polynomials in 4 variables $x, y, u, v$ is 9-dimensional and is spanned by $x^2 + y^2 - u^2 - v^2$, $x^2 - y^2$, $u^2 - v^2$, $xy$, $xu$, $xv$, $yu$, $yv$, $uv$. Hence, for $2 \leq n \leq 8$, a full quadratic harmonic map $f\colon S^3 \to S^n$ is given by

$$
\begin{aligned}
f(x,y,u,v) &= b_1x^2 + b_2y^2 + c_1u^2 + c_2v^2 \\
&\quad + d_1xy + d_2xu + d_3xv + d_4yu + d_5yv + d_6uv,
\end{aligned} \tag{1}
$$

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where the vectors $b_i, c_i, d_j \in \mathbf{R}^{n+1}$, $i = 1,2$, $j = 1,\ldots,6$, span $\mathbf{R}^{n+1}$ and $b_1 + b_2 + c_1 + c_2 = 0$. As $\|f(x,y,u,v)\|^2$ is a homogeneous polynomial of degree 4, the condition $\operatorname{im}(f) \subset S^n$ translates into

$$\|f(x,y,u,v)\|^2 = (x^2+y^2+u^2+v^2)^2, \tag{2}$$

which has to be satisfied for all $(x,y,u,v) \in \mathbf{R}^4$. Substituting (1) into (2) and expanding both sides, we obtain various orthogonality relations between $b_i, c_i, d_j \in \mathbf{R}^{n+1}$. For $n = 2$, a straightforward computation (using the vector cross product in $\mathbf{R}^3$) gives the general form of a full quadratic harmonic map $f\colon S^3 \to S^2$, namely, $f$ is equivalent to

$$\begin{aligned} f^{\varepsilon}_{\alpha,\beta}(x,y,u,v) = (&\cos\tfrac{\alpha}{2}(x^2+y^2-u^2-v^2) + 2\sin\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}(\varepsilon_3 xv + \varepsilon_4 yu) \\ &- 2\sin\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}(\varepsilon_2 xu + \varepsilon_5 yv), \\ &\sin\tfrac{\alpha}{2}(x^2-y^2-\cos\beta(u^2-v^2)) + 2\cos\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}(\varepsilon_2 xu - \varepsilon_5 yv) \\ &- 2\cos\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}(\varepsilon_3 xv - \varepsilon_4 yu) + 2\varepsilon_6\sin\tfrac{\alpha}{2}\sin\beta uv, \\ &-\sin\tfrac{\alpha}{2}\sin\beta(u^2-v^2) - 2\sin\tfrac{\alpha}{2}(\varepsilon_1 xy + \varepsilon_6\cos\beta uv) \\ &+ 2\cos\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}(\varepsilon_2 xu - \varepsilon_5 yv) + 2\cos\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}(\varepsilon_3 xv - \varepsilon_4 yu)), \end{aligned}$$

where $0 \le \alpha$, $\beta \le \pi$ and $\varepsilon = (\varepsilon_j)_{j=1}^6 \in \mathbf{Z}_2^6$ obeys the sign relations $\varepsilon_1\varepsilon_2\varepsilon_4 = -\varepsilon_1\varepsilon_3\varepsilon_5 = \varepsilon_2\varepsilon_3\varepsilon_6 = -\varepsilon_4\varepsilon_5\varepsilon_6 = 1$. For fixed $\varepsilon$, all $f^{\varepsilon}_{0,\beta}$, $0 \le \beta \le \pi$, are equivalent. Passing to the equivalence classes, we obtain a homeomorphic embedding of the triangle $[0,\pi]^2/\{0\} \times [0,\pi]$ into $\partial L^0$ (induced by $(\alpha,\beta) \to f^{\varepsilon}_{\alpha,\beta}$). By the sign relations, these 8 triangles (corresponding to the various $\varepsilon$) are easily seen to be pasted together along their edges to form two disjoint copies of the real projective plane $\mathbf{RP}^2$, each containing the Hopf map

$$f_2(x,y,u,v) = (x^2+y^2-u^2-v^2, 2(xu-yv), 2(xv+yu))$$

or its "dual"

$$f_2'(x,y,u,v) = (x^2+y^2-u^2-v^2, 2(xu+yv), 2(xv-yu)).$$

The symmetry group $G_{f_2} = \{U \in SO(4) \mid f_2 \circ U = V \circ f_2 \text{ for some } V \in O(3)\}$, being the isotropy subgroup of the point in $\partial L^0$ corresponding to $f_2$, is then at least 4-dimensional since the respective orbit is contained in a copy of $\mathbf{RP}^2$. On the other hand, $G_{f_2} \subset SO(4)$ is a proper subgroup since the Hopf map is not equivariant. It follows that $\dim G_{f_2} = 4$ and hence the $SO(4)$-orbit corresponding to $f_2$ should coincide with a copy of $\mathbf{RP}^2$. Passing to $O(4)$ we recover the other copy and (i) follows.

For (ii), we have to show that there is no system of vectors $b_i, c_i, d_j \in \mathbf{R}^4$, $i = 1,2$, $j = 1,\ldots,6$, spanning $\mathbf{R}^4$, such that they satisfy the orthogonality relations equivalent to (2). This can be done by tedious but elementary computation separating the cases $\dim\operatorname{span}\{b_i, c_i\}_{i=1}^2 = 1, 2$ or 3. (Note that in the last case, it is convenient to use the vector cross product on the 3-dimensional linear subspace spanned by $b_i$, $c_i$, $i = 1,2$.)

To prove (iii) needs an entirely different argument. Recall first that the parametrization of the equivalence classes of harmonic maps by $L^0$ is given by associating to the full quadratic harmonic map $f\colon S^3 \to S^n$, $2 \le n \le 8$, the symmetric matrix $A^t \cdot A - I_9 \in S^2(\mathbf{R}^9)$, where $A$ is the $(n+1) \times 9$-matrix defined by $f = A \circ f_{\lambda_2}$ with $f_{\lambda_2}\colon S^3 \to S^8$ a (fixed) standard minimal immersion. Then $E \subset S^2(\mathbf{R}^9)$ is the orthogonal complement of $W^0 = \operatorname{span}\{f_{\lambda_2}(x)^2 \mid x \in S^3\}$ and $L^0 = \{C - I_9 \in E \mid C \geq 0\}$. (Here $\geq$ stands for "symmetric and positive semidefinite".) In the same spirit, for fixed $f\colon S^3 \to S^n$ define $W_f^0 = \operatorname{span}\{f(x)^2 \mid x \in S^3\}$, $E_f = (W_f^0)^\perp \subset S^2(\mathbf{R}^{n+1})$ and $L_f^0 = \{C - I_{n+1} \in E_f \mid C \geq 0\}$. Then, the affine map $\phi\colon L_f^0 \to L^0$, defined by $\phi(C - I_{n+1}) = A^t \cdot C \cdot A - I_9$, injects $L_f^0$ onto a compact convex set $\overline{I}_f$. In the affine subspace spanned by $\overline{I}_f$, the interior $I_f$ of $\overline{I}_f$ is a convex body which contains the point corresponding to $f$. Thus the sets $I_f$, for various harmonic maps $f$, give rise to a subdivision of $L^0$ into disjoint convex sets. To show (iii), it is enough to give a series of full quadratic harmonic maps $f_n\colon S^3 \to S^n$, $4 \leq n \leq 8$, such that $\dim E_f (= \dim I_f) > 0$. First, let $f_6\colon S^3 \to S^6$ be defined by

$$\begin{aligned} f_6(x,y,u,v) = \Big(&\frac{1}{\sqrt{2}}(x^2+y^2-u^2-v^2), \frac{1}{\sqrt{2}}(x^2-y^2), \frac{1}{\sqrt{2}}(u^2-v^2), \\ &\sqrt{2}\,xy, \sqrt{3}(xu+yv), \sqrt{3}(yu-xv), \sqrt{2}\,uv\Big). \end{aligned}$$

Then $E_{f_6} \cong \mathbf{R}^3$ with $\overline{I}_{f_6}$ isomorphic to the finite (straight) cone in $\mathbf{R}^3$ with vertex $(1,0,0)$ and base circle of center $(-1,0,0)$ and radius 2. The origin corresponds to $f_6$; $(-1,0,0)$ corresponds to a full harmonic map $f_5\colon S^3 \to S^5$ with $\dim E_{f_5} = 2$ and the points on the (open) edges of the cone correspond to full harmonic maps $f_4\colon S^3 \to S^4$ with $\dim E_{f_4} = 1$. (Note that $f_5(x,y,u,v) = (x^2-y^2, u^2-v^2, 2xy, \sqrt{2}(xu+yv), \sqrt{2}(yu-xv), 2uv)$ and for $f_4$ one can also take the harmonic map obtained by applying the Hopf-Whitehead construction to the real tensor product $\otimes\colon \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^4$ [**3**].) Finally, define $f_8 = f_{\lambda_2}\colon S^3 \to S^8$ and $f_7\colon S^3 \to S^7$ by

$$f_7(x,y,u,v) = (x^2-y^2, u^2-v^2, 2xy, \sqrt{2}xu, \sqrt{2}xv, \sqrt{2}yu, \sqrt{2}yv, 2uv).$$

Then, $\dim E_{f_8} = 10$ and $\dim E_{f_7} = 5$ which completes the proof.

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Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210

*Current address*: Department of Mathematics, Rutgers University, Camden, New Jersey 08102