PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 2, February 1988

SMOOTH CONVEX t-NORMS DO NOT EXIST

C. ALSINA AND M. S. TOMAS

(Communicated by Richard R. Goldberg)

ABSTRACT. We show that smooth convex t-norms do not exist.

Let T be a *t-norm*, i.e., a binary operation on $[0,1]$ which is associative, commutative, nondecreasing in each place, and has 1 as a unit element. A t-norm T is *convex* if, for all λ in $[0,1]$ and for all a,b,c,d in $[0,1]$, T satisfies:

$$T(\lambda a + (1-\lambda)b, \lambda c + (1-\lambda)d) \leq \lambda T(a,c) + (1-\lambda)T(b,d).$$

Convex t-norms play a fundamental role in some studies of products of probabilistic metric spaces (see [**2, 6**]). An example of continuous convex t-norm is $W(x,y) = \mathrm{Max}(x+y-1,0)$ and a discontinuous example is $Z(x,y) = 0$ whenever (x,y) is in $[0,1) \times [0,1)$ and $Z(x,y) = W(x,y)$ otherwise.

All other examples of continuous convex t-norms are closely related to W. Consequently, it is natural to ask whether smooth convex t-norms can exist. Our aim in this paper is to show that they cannot. To this end we begin with the following

LEMMA 1. *If T is a continuous convex t-norm, then*: (i) $T \leq W$; (ii) T *admits a representation of the form* $T(x,y) = f^{(-1)}(f(x)+f(y))$, *where f (the additive generator of T) is a continuous strictly decreasing function from* $[0,1]$ *onto* $[0,1]$ *such that* $f(0)=1$, $f(1)=0$, *and* $f^{(-1)}$ *is the pseudo-inverse of f, i.e.,* $f^{(-1)} = f^{-1}$ *on* $[0,1]$ *and* $f^{(-1)}(x) = 0$, *whenever* $x \geq 1$; (iii) f *is concave on* $[f^{-1}(\frac{1}{2}),1]$.

PROOF. Since T is a convex t-norm, for any a,b and λ in $[0,1]$ we have

$$T(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) \leq \lambda T(a,b) + (1-\lambda)T(b,a) = T(a,b). \tag{1}$$

Suppose x,y in $[0,1]$ are such that $x+y>1$. Then substituting $a = x+y-1$, $b=1$, and $\lambda = (1-x)/(2-x-y)$ in (1) yields $T(x,y) \leq T(x+y-1,1) = x+y-1$, from which (i) follows. The representation of T follows from the theorem of Ling [**5**]. Finally, the substitution $\lambda = \frac{1}{2}$ into (1) yields $T(\frac{1}{2}(a+b), \frac{1}{2}(a+b)) \leq T(a,b)$. Using the representation, for any a,b in $[f^{-1}(\frac{1}{2}),1]$ we have $(a+b)/2 \geq f^{-1}(\frac{1}{2})$ and $2f((a+b)/2) < 1$. Therefore $f^{-1}(2f((a+b)/2)) > 0$ and we obtain

$$0 < f^{-1}\left(2f\left(\frac{a+b}{2}\right)\right) = T\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \leq T(a,b) = f^{-1}(f(a)+f(b)),$$

whence $f((a+b)/2) \geq \frac{1}{2}(f(a)+f(b))$, i.e., f is concave on $[f^{-1}(\frac{1}{2}),1]$.

DEFINITION 1. A t-norm T representable in the form

$$T(x,y) = f^{(-1)}(f(x)+f(y))$$

Received by the editors April 2, 1986 and, in revised form, September 15, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 39C05, 26B25.

will be called *smooth* if f' and f'' exist on $(0,1)$, $f'(x) \neq 0$ for all x in $(0,1)$, $\lim_{x\to 0^+} x \cdot f'(x) = -\infty$, and $f''(x)$ is bounded on an arbitrary left-neighborhood of 1 of the form $(1-\varepsilon, 1)$.

Thus for smooth t-norms T the first and second partial derivatives of T exist on $(0,1)\times(0,1)$.

LEMMA 2. *If $T(x,y) = f^{(-1)}(f(x)+f(y))$ is a smooth convex t-norm, then*

$$\lim_{k\to 1^-} \frac{f''(k)}{f'(k)^2} = 0.$$

PROOF. Since f'' is assumed to be bounded on a neighborhood of 1, we will prove that $\lim_{k\to 1^-} f'(k) = -\infty$. Choose a sequence $(y_n)_{n\in N}$ converging to 0 so that $(1-f(y_n))/y_n > n$. (This is possible because $\lim_{x\to 0^+} f'(x) = -\infty$.) Let $x_n = f^{-1}(1-f(y_n))$. Since $f(x_n)+f(y_n) = 1$ and $T \le W$ we have

$$0 = T(y_n, 1-y_n) = f^{(-1)}(1-f(x_n)+f(1-y_n)),$$

i.e., $1-f(x_n)+f(1-y_n) \ge 1$. Hence $1-x_n \le y_n$ and we have

$$\frac{f(x_n)}{1-x_n} \ge \frac{f(x_n)}{y_n} = \frac{1-f(y_n)}{y_n} > n.$$

It follows that

$$\lim_{x\to 1^-} f'(x) = \lim_{n\to\infty} \frac{f(x_n)}{x_n - 1} = -\infty.$$

LEMMA 3. *If g is a function from $(0,1)$ into $\mathbf{R}$ satisfying*
(i) *$g(u) \le 0$ whenever u is in $(0,\frac{1}{2}]$;*
(ii) *$g(u) \le g(u+v)$ for all u,v in $(0,1)$ with $u+v \le 1$;*
(iii) *$g(u)\cdot g(v) \ge (g(u)+g(v))\cdot g(u+v)$ for all u,v in $(0,1)$ with $u+v\le 1$;*
(iv) *$\lim_{h\to 0^+} g(h)$ exists*
then $g(x) = 0$ for all x in $(0,1)$.

PROOF. Given any x in $(0,\frac{1}{2}]$ the substitution $u = v = x$ into (iii) yields $g(x)^2 \ge 2g(x)g(2x)$. If $g(x) < 0$ then $g(x) \le 2g(2x)$ and by induction

$$g(2x) \ge \frac{1}{2}g(x) \ge \frac{1}{2^{(n+1)}}\cdot g\left(\frac{x}{2^n}\right),$$

whence by (iv)

$$g(2x) \ge \frac{1}{2}g(x) \ge \lim_{n\to\infty} \frac{1}{2^{n+1}}\cdot g\left(\frac{x}{2^n}\right) = 0,$$

i.e., $g(x) \ge 0$ in contradiction with $g(x) < 0$. Therefore by (i) we have $g(x) = 0$ whenever x is in $(0,\frac{1}{2}]$. Now for any x in $(0,1)$ we have by (ii), $g(x/2) = 0 \le g(x)$. Consequently for any u,v in $(0,1)$ with $u+v\le 1$ we will have by (iii) and (ii):

$$g(u)\cdot g(v) \ge (g(u)+g(v))\cdot g(u+v) \ge (g(u)+g(v))g(u).$$

Thus $0 \ge g(u)^2$ and we can conclude $g(u) = 0$ for all u in $(0,1)$. We remark that condition (iv) is essential because there are nonzero functions such as $g(x) = -1/x$ that satisfy (i), (ii), and (iii).

THEOREM 1. *Smooth convex t-norms cannot exist.*

PROOF. Let T be a continuous convex t-norm. By Lemma 1, T can be represented in the form

$$T(x,y) = f^{(-1)}(f(x) + f(y)), \tag{2}$$

where f is concave on $[f^{-1}(\frac{1}{2}), 1]$. If T is smooth and convex then it is well known [4] that the following conditions hold:

$$\frac{\partial^2 T}{\partial x^2}(u,v) \geq 0 \tag{3}$$

and

$$\frac{\partial^2 T}{\partial x^2}(u,v) \cdot \frac{\partial^2 T}{\partial y^2}(u,v) - \left[\frac{\partial^2 T}{\partial x \partial y}(u,v)\right]^2 \geq 0 \tag{4}$$

for all u, v in $(0,1)$. Using (2), after some computations we have that for any x, y in $(0,1)$ with $f(x) + f(y) < 1$ the inequalities (3) and (4) become

$$\frac{f''(x)}{f'(x)^2} \leq \frac{f''(T(x,y))}{f'(T(x,y))^2}, \tag{5}$$

and

$$\frac{f''(x)f''(y)}{f'(x)^2 f'(y)^2} \geq \frac{f''(x)f''(T(x,y))}{f'(x)^2 f'(T(x,y))^2} + \frac{f''(y)f''(T(x,y))}{f'(y)^2 f'(T(x,y))^2}. \tag{6}$$

Let $f(x) = u$ and $f(y) = v$. Then for u, v in $(0,1)$ with $u + v < 1$, we have

$$\frac{f''(f^{-1}(u))}{f'(f^{-1}(u))^2} \leq \frac{f''(f^{-1}(u+v))}{f'(f^{-1}(u+v))^2} \tag{7}$$

and

$$\begin{aligned} &\frac{f''(f^{-1}(u))}{f'(f^{-1}(u))^2} \cdot \frac{f''(f^{-1}(v))}{f'(f^{-1}(v))^2} \\ &\quad \geq \left[\frac{f''(f^{-1}(u))}{f'(f^{-1}(u))^2} + \frac{f''(f^{-1}(v))}{f'(f^{-1}(v))^2}\right] \cdot \frac{f''(f^{-1}(u+v))}{f'(f^{-1}(u+v))^2}. \end{aligned} \tag{8}$$

Define g from $(0,1)$ into $\mathbf{R}$ by $g(z) = f''(f^{-1}(z))/f'(f^{-1}(z))^2$. Obviously g is well defined and by Lemma 2 we have

$$\lim_{k \to 1^-} \frac{f''(k)}{f'(k)^2} = \lim_{h \to 0^+} g(h) = 0.$$

Since f is concave on $[f^{-1}(\frac{1}{2}), 1]$ we have

$$g(u) \leq 0 \quad \text{whenever } u \text{ is in } (0, \tfrac{1}{2}]; \tag{9}$$

and by (7) and (8) for all u, v in $(0,1)$ with $u + v < 1$ we obtain

$$g(u) \leq g(u+v) \tag{10}$$

and

$$g(u) \cdot g(v) \geq (g(u) + g(v)) \cdot g(u+v). \tag{11}$$

Now we can apply Lemma 3 and conclude that g must be the zero function, i.e., that $f''(f^{-1}(z)) = 0$ for all z in $(0,1)$. This in turn implies that f must have the form $f(x) = ax + b$ for some constants a, b and contradicts the assumption that $\lim_{x\to 0^+} f'(x) = -\infty$.

ACKNOWLEDGMENTS. The authors thank Professor M. J. Frank (Illinois Institute of Technology, Chicago) for his interesting remarks concerning this paper and the referee for some helpful comments.

BIBLIOGRAPHY

1. J. Aczél, *Lectures on functional equations and their applications*, Academic Press, New York and London, 1966.
2. C. Alsina, *On countable products of probabilistic metric spaces and algebraic convexifications*, Pacific J. Math. **76** (1978), 291–300.
3. C. Alsina and B. Schweizer, *The countable product of probabilistic metric spaces*, Houston J. Math. **9** (1983), 303–310.
4. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1934.
5. C. H. Ling, *Representation of associative functions*, Publ. Math. Debrecen **12** (1965), 189–212.
6. B. Schweizer and A. Sklar, *Probabilistic metric spaces*, Elsevier, New York, 1983.

DEPARTAMENT MATEMÀTIQUES I ESTADÍSTICA, ESCOLA TÉCNICA SUPERIOR D'ARQUITECTURA, UNIVERSITAT POLITÉCNICA DE CATALUNYA, AVDA DIAGONAL, 649, 08028 BARCELONA, SPAIN