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ON FORBIDDEN MINORS FOR $GF(3)$

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ABSTRACT. A new, surprisingly simple proof is given of the finiteness of the set of matroids minor-minimally not representable over $GF(3)$. It is, in fact, proved that every such matroid has rank or corank at most 3.

Introduction. For q a prime power we denote by $\mathcal{L}(q)$ the class of matroids representable over $GF(q)$, and by $\mathcal{F}(q)$ the class of matroids which are minor-minimally not in $\mathcal{L}(q)$. It was conjectured first by Rota [**R**] that $\mathcal{F}(q)$ is finite for each q. This had been proved earlier by Tutte [**T2**] for $q = 2$, and has since been proved in a number of ways for $q = 3$ (R. Reid, unpublished circa 1972, or [**S**, **B**, **T1**, **K**]); but for larger q the conjecture remains open.

Most of the above mentioned proofs proceed by actully determining the set $\mathcal{F}(q)$, but, as remarked in [**K1**], nothing so precise is likely to be possible for general q. Our purpose here is to present a surprisingly simple proof of the finiteness of $\mathcal{F}(3)$, specifically proving

THEOREM. *If $M \in \mathcal{F}(3)$ then either M or M^* has rank at most* 3.

(Recall $M \in \mathcal{F}(3) \Leftrightarrow M^* \in \mathcal{F}(3)$.) Of course this reduces the precise determination of $\mathcal{F}(3)$ to the determination of its rank 3 members, a problem easily settled by ad hoc arguments.

For matroid background we refer to Welsh [**W**], from which our notation differs in that we use $r(A)$ and $\bar{A}$ for the rank and closure of a set A.

1. A Lemma. To a large extent our proof parallels the easier part of the argument of [**S**]. The new idea, which eliminates virtually all of the hard work of that paper, is a timely application of the following simple fact.

LEMMA. *Let M be a connected simple matroid of rank $r \geq 2$ on a set S and let*

$$X = \{x \in S\colon M/x \text{ is disconnected}\}.$$

Then

(a) $|X| \leq r - 2$, *and*

(b) *if $|X| = r - 2$ then there are lines $l_0, \ldots, l_{r-2}$ and an ordering $x_1, \ldots, x_{r-2}$ of X such that*

(i) $|l_i| \geq 3$,

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(ii) $S = \bigcup_{i=0}^{r-2} l_i$, *and*

(iii) *for* $i \geq 1$, $l_i \cap \overline{(l_0 \cup \cdots \cup l_{i-1})} = \{x_i\}$.

(We remark that condition (i) in (b) is a consequence of conditions (ii) and (iii) and the connectivity of M.)

The proof will follow a few preliminaries. Let M be as in the hypotheses of the Lemma. We define a *bisection* of M at $x \in S$ to be a pair (A, B) of flats of M satisfying

$$A \cup B = S \neq A, B, \qquad A \cap B = \{x\}$$

and

$$r(A) + r(B) = r(S) + 1.$$

If M is connected the following are easily verified.

(1.1) If (A, B) is a bisection of M then $M|A$ and $M|B$ are connected.

(1.2) M/x is disconnected if and only if there is a bisection of M at x.

(1.3) If (A, B) is a bisection then every circuit meeting both $A\backslash x$ and $B\backslash x$ is of the form $C \triangle D$ with $C \subseteq A$ and $D \subseteq B$ circuits containing x (and $\triangle$ denoting symmetric difference).

Suppose now that M is connected, that (A, B) is a bisection at x, and that (A', B') is a bisection at some $x' \in A\backslash B$. From (1.1) and (1.3) we have immediately

(1.4) $(A' \cap A, B' \cap A)$ is a bisection of A.

(Note e.g. $A' \cap A \neq \{x'\}$ since this would force $x' \in \bar{B}$.) Moreover

(1.5) only one of A', B' meets B.

For suppose this is false. By (1.1) there is a circuit $C \subseteq B$ meeting both $A' \cap B$ and $B' \cap B$, and by (1.3) $C = C_1 \triangle C_2$ where $C_1 \subseteq A'$, $C_2 \subseteq B'$ and $C_1 \cap C_2 = \{x'\}$. But then $C_1\backslash\{x'\} \subseteq C \subseteq B$ and this implies that x' is in the span of B, a contradiction. □

Proof of Lemma. We proceed by induction on r, the case $r = 2$ being trivial. Assume, then, that $r \geq 3$ and $|X| \geq r - 2$. Let (A, B) be a bisection of M with A minimal at some $x \in X$. By (1.5) we must have

(1.6) $X \subseteq B$.

But now setting $M' = M|B$ and $X' = X\backslash\{x\}$ we apply (1.4) (with A and B reversed) and (1.2) to find that

(1.7) M'/x' is disconnected for each $x' \in X'$.

But then since M' is connected (by (1.1)) with $r(M') \leq r - 1$ and $|X'| \geq r - 3$, our inductive hypothesis implies that equality holds in both places (note this already proves (a)), that A is a line, that

$$X' = \{x' \colon M'/x' \text{ is disconnected}\},$$

and that we can find lines $l_0, \ldots, l_{r-3}$ of M' and an ordering $x_1, \ldots, x_{r-3}$ of X' satisfying the conditions of (b). The proof of (b) is now completed by taking $l_{r-2} = A$ and $x_{r-2} = x$. □

2. Proof of the Theorem. Suppose henceforth that M is a rank r matroid on S, $M \in \mathcal{F}(3)$.

It is easy to see that any member of any $\mathcal{F}(q)$ must be 3-connected, and so the matroid as well as all its single element deletions and contractions are connected. We recall Lemma 2.3 of [S].

(2.1) *If M is connected and $M\backslash x$ and M/x are connected for each $x \in S$, then either $M = U_4^2$ or there exist $a, b \in S$ such that one of $M\backslash\{a,b\}$, $M/\{a,b\}$ is connected.*

(As usual U_4^2 is the four element matroid whose circuits are its sets of size 3.)

Since $U_4^2 \notin \mathcal{F}(3)$, and since our Theorem is self-dual, we may suppose that

(2.2) *There exist $a, b \in S$ such that $M\backslash\{a,b\}$ is connected.*

We attempt to represent M over $GF(3)$ by starting with a representation

$$\phi\colon S\backslash\{a,b\} \to V = V_r(3)$$

of $M\backslash\{a,b\}$, and extending ϕ to a and b so that $\phi|_{S\backslash b}$ and $\phi|_{S\backslash a}$ are representations of $M\backslash b$ and $M\backslash a$. (This is possible since, as shown in [**BL**], $GF(3)$-representations are projectively unique.) Let M' be the matroid on S represented by ϕ. Then M and M' are related as follows.

(2.3) M and M' are distinct connected matroids on a common set S,

(2.4) $M\backslash a = M'\backslash a$ and $M\backslash b = M'\backslash b$,

(2.5) $M\backslash\{a,b\}$ $(= M'\backslash\{a,b\})$ is connected.

PROPOSITION. *If matroids M and M' satisfy* (2.3)–(2.5) *then at most one of them is in* $\mathcal{L}(3)$.

PROOF. Suppose both are in $\mathcal{L}(3)$, and let r denote their common rank. By the uniqueness of $GF(3)$-representations [**BL**] any representation

$$\phi\colon S\backslash\{a,b\} \to V$$

(V an r-dimensional $GF(3)$-vector space) of $M\backslash\{a,b\} = M'\backslash\{a,b\}$ can be extended to representations $\psi\colon S \to V$ and $\psi'\colon S \to V$ of M and M' respectively. But up to scaling there is only one choice for $\psi(a)$ which makes $\psi|_{S\backslash b}$ a representation of $M\backslash b$, and similarly for $\psi(b)$. (This follows from the uniqueness of representations together with Kantor's observation [**K2**, Lemma 7] that if K is a field with no nonidentity automorphisms and X a connected spanning subset of some projective space over K, then no nonidentity automorphism of the space fixes X elementwise.) Of course the same discussion applies to M', so from (2.4) we deduce that ψ and ψ' are equal up to scaling, and therefore M and M' are equal. $\square$

In view of this result, our proof of the Theorem can be completed by showing

(2.6) *If M and M' are a minor-minimal pair of matroids satisfying* (2.3)–(2.5) *then their* (*common*) *rank is at most* 3.

PROOF. Let Z be a minimal set which is dependent in one of M, M' (say M) and independent in the other. Then as in the proof of Lemma 2.7 of [**S**], the minimality of M, M' implies that Z is unique and satisfies, among other things,

(2.7) $a, b \in Z$,

(2.8) Z is a circuit and a hyperplane of M,

(2.9) For every $z \in Z\backslash\{a,b\}$, $M\backslash\{a,b\}/\{z\}$ is disconnected.

Now suppose $r \geq 4$. Since $|Z| = r$ (by (2.8)), (2.9) and the Lemma imply that there exist $x, y \in Z\backslash\{a,b\}$ (take x, y to be x_1, x_2 as given by the Lemma) such that the closure of $\{x,y\}$ in M contains some third element. But this contradicts (2.8) and we conclude that $r \leq 3$. This completes the proof of the Theorem. $\square$

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