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A NOTE ON THE MINIMUM PROPERTY

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ABSTRACT. It is shown that a Banach space E has a strictly convex dual if and only if for every accretive operator $A \subset E \times E$ that satisfies the range condition at infinity, $\mathrm{cl}(R(A))$ has the minimum property.

The minimum property was introduced by Pazy in [**4**] and has turned out to be useful in the study of the asymptotic behavior of nonlinear semigroups in Banach spaces (see [**6**]). Our purpose in this note is to show that for a Banach space E the strict convexity of its dual E^* is a necessary and sufficient condition for E to have the following property: for every accretive operator $A \subset E \times E$ that satisfies the range condition at infinity, $\mathrm{cl}(R(A))$ has the minimum property (see the definitions below). The sufficiency part is known [**5**], so we only prove the reverse implication. In [**1**] it was shown that the smoothness of E is necessary; here we prove that in fact it is necessary that E^* be strictly convex, so we are able to remove the reflexivity condition which was necessary for Theorem 1 in [**1**] (recall that a nonreflexive Banach space E may be smooth while E^* is not strictly convex; see [**2**]).

Let E be a real Banach space and let I denote the identity operator. Recall that a subset A of $E \times E$ with domain $D(A)$ and range $R(A)$ is said to be accretive if $|x_1 - x_2| \le |x_1 - x_2 + r(y_1 - y_2)|$ for all $[x_i, y_i] \in A$, $i = 1, 2$, and $r > 0$.

For example, if T is a nonexpansive mapping, then $I - T$ is an accretive operator. We denote the closure of a subset D of E by $\mathrm{cl}(D)$, its closed convex hull by $\mathrm{clco}(D)$ and its distance from a point x in E by $d(x, D)$. We shall say that A satisfies the range condition at infinity if $\liminf_{t\to\infty} d(0, R(I + tA))/t = 0$. A closed subset D of E is said to have the minimum property if $d(0, \mathrm{clco}(D)) = d(0, D)$. For a Banach space E we set $S(E) = \{x \in E;\ |x| = 1\}$. E is said to be strictly convex if $x, y, (x + y)/2 \in S(E)$ implies that $x = y$. We begin with a simple Lemma.

LEMMA. *E^* is strictly convex if and only if*

$$(1) \qquad x^*, y^* \in S(E^*),\ \{x_n\}_{n=1}^\infty \subset S(E) \text{ and } \lim_{n\to\infty} (x_n, x^*) = \lim_{n\to\infty} (x_n, y^*) = 1$$

imply $x^* = y^*$.

PROOF. Suppose that E^* is strictly convex and that there are $x^*, y^* \in S(E^*)$ and a sequence $\{x_n\}_{n=1}^\infty \subset S(E)$ such that $\lim_{n\to\infty}(x_n, x^*) = \lim_{n\to\infty}(x_n, y^*) = 1$. We then also have $\lim_{n\to\infty}(x_n, (x^* + y^*)/2) = 1$ and so $|(x^* + y^*)/2| = 1$. Since E^* is strictly convex this implies that $x^* = y^*$. Conversely, suppose that (1) holds in E. Let $x^*, y^* \in S(E^*)$ be such that $(x^* + y^*)/2 \in S(E^*)$. We choose a sequence $\{x_n\}_{n=1}^\infty \subset S(E)$ such that $\lim_{n\to\infty}(x_n, (x^* + y^*)/2) = 1$. This implies

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that $\lim_{n\to\infty}(x_n, y^*) = \lim_{n\to\infty}(x_n, x^*) = 1$ and by (1) we must have $x^* = y^*$, so E^* is strictly convex.

Next we state and prove our main theorem.

THEOREM. *A Banach space E has a strictly convex dual if and only if* $\mathrm{cl}(R(A))$ *has the minimum property for all accretive $A \subset E \times E$ that satisfy the range condition at infinity.*

PROOF. As mentioned above, we only have to prove the necessity part. We assume that E^* is not strictly convex. So by the Lemma there are $x^*, y^* \in S(E^*)$, $x^* \neq y^*$ and $\{x_n\}_{n=1}^{\infty} \subset S(E)$ such that $\lim_{n\to\infty}(x_n, x^*) = \lim_{n\to\infty}(x_n, y^*) = 1$. By choosing a subsequence of the $\{x_n\}_{n=1}^{\infty}$ if necessary we may assume that $\forall n \geq 1$, $(x_n, x^*) \leq (x_n, y^*)$ and that

$$\sum_{n=1}^{\infty}(1 - (x_n, x^*)) \leq \tfrac{1}{2}. \tag{2}$$

Let $\{\gamma(t)\}_{t\geq 0}$ be a piecewise linear curve starting at 0 such that its derivative on $[i-1, i)$ is x_i for $i \geq 1$. By (2) we have $t \geq (\gamma(t), y^*) \geq (\gamma(t), x^*) \geq t - \frac{1}{2}$ $\forall t \geq 0$.

We set $C = \{x \in E;\ 0 \leq (x, x^*) \leq (x, y^*)\}$. It is easy to see that C is closed and convex and that $\{\gamma(t)\}_{t\geq 0} \subset C$. We define a mapping $T\colon C \to C$ by $Tx = \gamma((x, x^*)+1)$. $\forall x \in C$. For $x, z \in C$ we have $|Tx - Tz| \leq |(x, x^*)+1-((z, x^*)+1)| \leq |x - z|$ so T is nonexpansive. In addition, for every $x \in C$:

$$|x - Tx| \geq (Tx - x, x^*) = (\gamma((x, x^*) + 1), x^*) - (x, x^*) \geq \tfrac{1}{2}. \tag{3}$$

Since $x^* \neq y^*$ there is $y \in E$ such that $(y, y^*) < 0 < (y, x^*)$. We define $u = -y/(y, x^*)$, $v = -y/(y, y^*)$ and then $D = C \cup \{u, v\}$. We extend T to D by $Tu = Tv = 0$. For $x \in C$ we have

$$|Tx - Tu| = |Tx| = |\gamma((x, x^*) + 1)| \leq (x, x^*) + 1 = (x - u, x^*) \leq |x - u|,$$
$$|Tx - Tv| = |\gamma((x, x^*) + 1)| \leq (x, x^*) + 1 \leq (x, y^*) + 1 = (x - v, y^*) \leq |x - v|.$$

So $T\colon D \to D$ is a nonexpansive mapping. We also have $|u - Tu|$, $|v - Tv| \geq 1$. Combining this with (3) we conclude that the accretive operator $A = I - T$ with domain D satisfies $d(0, R(A)) \geq \frac{1}{2}$. Since T maps C into itself, we have for all $r > 0$, $C \subset R(I + rA)$ and the range condition at infinity is certainly satisfied for A. But $\mathrm{cl}(R(A))$ does not possess the minimum property because

$$0 = [(y, x^*)u + (-(y, y^*))]/[(y, x^*) + (-(y, y^*))] \in \mathrm{co}(R(A)).$$

We note that the construction above is related to the one given in [**3**].

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