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A STRUCTURE THEOREM FOR DISCONTINUOUS DERIVATIONS OF BANACH ALGEBRAS OF DIFFERENTIAL FUNCTIONS

VIET NGO

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ABSTRACT. Let $D\colon C^n[0,1] \to \mathcal{M}$ be a derivation from the Banach algebra of n times continuously differentiable functions on $[0,1]$ into a Banach $C^n[0,1]$-module $\mathcal{M}$. If D is continuous then it is completely determined by $D(z)$ where $z(t) = t$, $0 \leq t \leq 1$. For the case when D is discontinuous we show that $D(f)$ is determined by $D(z)$ for all f in an ideal $\mathcal{T}(D)^2$ of $C^n[0,1]$ where its closure $\overline{\mathcal{T}(D)^2}$ is of finite codimension.

0. Introduction. In 1978 Bade and Curtis [**2**] constructed a derivation from $C^1[0,1]$ into $L_p(0,1)$ which is discontinuous on every dense subalgebra. The strickingly high degree of discontinuity of the derivation in that example challenged the investigation on the existence and the characterization of derivations of $C^n[0,1]$ with similar properties. In this paper we present a key theorem on discontinuous derivations of $C^n[0,1]$ which is a stepping stone in the quest of determining the structure of all discontinuous derivations of $C^n[0,1]$.

1. Preliminaries. Let $C^n[0,1]$ denote the algebra of all complex valued functions on $[0,1]$ which have n continuous derivatives. It is well known that $C^n[0,1]$ is a Banach algebra under the norm

$$\|f\| = \max_{t\in[0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!}$$

whose structure space is $[0,1]$. We will need a characterization of the squares of the closed primary ideals with finite codimension in $C^n[0,1]$. We use the notation

$$M_{n,k}(t_0) = \{f \in C^n[0,1] | f^{(j)}(t_0) = 0;\ j = 0,1,\ldots,k\}.$$

These are precisely the closed ideals of finite codimension contained in the maximal ideal $M_{n,0}(t_0)$ of functions vanishing at t_0. Writing $M_{n,k}$ for $M_{n,k}(0)$ and setting $z(t) = t$, $0 \leq t \leq 1$, we have

1.1 THEOREM. *Let n be a positive integer. Then*
(i) $M_{n,0}^2 = zM_{n,0} = \{f | f(0) = f'(0) = 0$ *and* $f^{(n+1)}(0)$ *exists*$\}$,
(ii) $M_{n,k}^2 = z^{k+1}M_{n,k}$, $1 \leq k \leq n-1$,
(iii) $M_{n,n}^2 = z^n M_{n,n}$.

Part (i) is from [**1**, Example 3]. Part (ii) is due to Dales and McClure [**3**, Theorem 3.1]. The proof of part (iii) can be found in [**2**].

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The squares of the closed primary ideals $M_{n,k}(t_0)$ at other points t_0 in $[0,1]$ are given by exactly similar formulas, where z is replaced by $z - t_0$.

A Banach $C^n[0,1]$-module is a Banach space $\mathcal{M}$ together with a continuous homomorphism $\rho\colon C^n[0,1] \to \mathcal{B}(\mathcal{M})$. A *derivation*, or a *module derivation* of $C^n[0,1]$ into $\mathcal{M}$ is a linear map $D\colon C^n[0,1] \to \mathcal{M}$ which satisfies the identity

$$D(fg) = \rho(f)D(g) + \rho(g)D(f).$$

We shall be interested in the form taken by discontinuous derivations. To measure the discontinuity of a derivation D, one introduces *separating space* $\mathcal{S}(D)$. This is the subspace of $\mathcal{M}$ defined by

$$\mathcal{S}(D) = \{m \in \mathcal{M} | \exists \{f_k\} \subseteq C^n[0,1],\ f_k \Rightarrow 0 \text{ and } D(f_k) \Rightarrow m\}.$$

It is easily checked that $\mathcal{S}(D)$ is a closed submodule of $\mathcal{M}$, and the derivation D is continuous if and only if $\mathcal{S}(D) = (0)$. The *continuity ideal* for a derivation $D\colon C^n[0,1] \to \mathcal{M}$ is

$$\mathcal{T}(D) = \{f \in C^n[0,1] | \rho(f)\mathcal{S}(D) = (0)\}.$$

$\mathcal{T}(D)$ is a closed ideal in $C^n[0,1]$. It is proved in [**1**, Theorem 3.2] that

$$\mathcal{T}(D) = \{f \in C^n[0,1] | D_f \text{ is continuous}\},$$

where $D_f(\cdot) = \rho(f)D(\cdot)$.

The hull F of $\mathcal{T}(D)$ is called the *singularity set* for D. If D is a derivation from $C^n[0,1]$ with singularity set F, then F is finite and $\mathcal{T}(D) \supseteq \bigcap_{t\in F} M_{n,n-1}(t)$. Moreover we can decompose D into a finite sum of derivations whose singularity sets consist of exactly one point [**2**, Theorems 1.2 and 3.2]. Throughout this paper we shall assume that a discontinuous derivation has the point zero for its singularity set.

If $D\colon C^n[0,1] \to \mathcal{M}$ is a derivation, we have

$$D(p(z)) = \rho(p'(z))D(z), \qquad p \in P,$$

where P is the dense subalgebra of polynomials in z. If D is continuous, it is completely determined by this formula. Thus a continuous derivation D is uniquely determined by the vector $D(z)$. We shall see that if D is a discontinuous derivation, it is still determined by $D(z)$ on the square of the continuity ideal. But first, we need to define the notion of the differential subspace of a Banach $C^n[0,1]$-module, a concept first introduced by Kantorovitz who named it "semisimplicity manifold" [**4**, **5**]. Let $\mathcal{M}$ be a Banach $C^n[0,1]$-module. *The differential subspace* is the set W of all vectors m such that the map $p \to \rho(p')m$ is continuous on P. We quote the following results from [**2**].

1.2 THEOREM. *Let $\mathcal{M}$ be a $C^n[0,1]$-module. A vector m lies in the differential subspace W if and only if the map $\rho \to \rho(p)m$ is continuous for the $C^n[0,1]$-norm on P. For $m \in W$, define*

$$|||m||| = \sup\left\{\|\rho(p)m\| \;\middle|\; \|p\|_{n-1} = 1\right\}.$$

then

(1) $\|m\| \leq |||m|||$, $m \in W$,

(2) *W is a Banach space under the norm $|||\cdot|||$,*

(3) *W is a $C^{n-1}[0,1]$-module. There exists a unique continuous homomorphism $\gamma\colon C^{n-1}[0,1] \to W$ such that*

$$\gamma(p)m = \rho(p)m, \qquad m \in W,\ p \in P.$$

Since $D(f) = \gamma(f')D(z)$ for every continuous derivation $D\colon C^n[0,1] \to \mathcal{M}$ [**2**, Theorem 4.5], a computation of γ, given $\mathcal{M}$ and $\rho\colon C^n[0,1] \to \mathcal{B}(\mathcal{M})$, will give us an explicit structure of continuous derivations of $C^n[0,1]$ into $\mathcal{M}$.

A nontrivial derivation $D\colon C^n[0,1] \to \mathcal{M}$ is called *singular* if D vanishes on P (equivalently $D(z) = 0$). A singular derivation is necessarily discontinuous. A derivation D is decomposable if D can be expressed in the form $D = E + F$, where E is continuous and F is singular. Such a splitting is unique. It was shown in [**2**] that a derivation $D\colon C^n[0,1] \to \mathcal{M}$ is decomposable if and only if $D(z) \in W$. If D is decomposable and $D = E + F$, then its singular part F vanishes also on $\mathcal{T}(D)^2$.

An indecomposable derivation is a derivation which is not decomposable. As an application we shall determine the structure of all derivations from $C^1[0,1]$ into $L_p(0,1)$, $1 \le p < \infty$, where the module action is defined by

$$(\rho(f)x)(t) = f(t)x(t) - \int_0^t f'(s)x(s)\,ds, \qquad f \in C^1[0,1],\ x \in L_p(0,1).$$

2. Discontinuous derivations of $C^n[0,1]$. In this section we present a structure theorem for continuous and discontinuous derivations of $C^n[0,1]$ into an arbitrary $C^n[0,1]$-module. But first we need the following lemma.

2.1 LEMMA. *Fix $n \ge 1$, and equip $M_{n,n-1}$ with the $C^n[0,1]$-norm. The map $f \to f/z$ from $M_{n,n-1}$ to $C^{n-1}[0,1]$ is continuous.*

PROOF. We show that $\|(z^{-1}f)^{(n-1)}\|_\infty \le n!2^{n-1}\|f\|_n$. By the Leibnitz formula

$$\begin{aligned}(z^{-1}f)^{(n-1)} &= \sum_{j=0}^{n-1}\binom{n-1}{j}(z^{-1})^{(j)}f^{(n-1-j)} \\ &= \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j j! z^{-j-1}f^{(n-1-j)}.\end{aligned}$$

Thus

$$(z^{-1}f)^{(n-1)}(t) = \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j j!\frac{1}{t^{j+1}}\int_0^t\int_0^{r_1}\cdots\int_0^{r_j} f^{(n)}(s)\,ds\,dr_j\cdots dr_1,$$

so that

$$\begin{aligned}|(z^{-1}f)^{(n-1)}(t)| &\le \sum_{j=0}^{n-1}\binom{n-1}{j}j!\frac{1}{t^{j+1}}(n!\|f\|_n)\frac{1}{j!}t^{j+1} \\ &= n!\|f\|_n\sum_{j=0}^{n-1}\binom{n-1}{j}.\end{aligned}$$

Thus $\|z^{-1}f\|_{n-1} \le K\|f\|_n$ for some $K > 0$.

2.2 REMARK. The map $f \to f/z$ is a continuous map from $M_{n,n-1}$ to $M_{n-1,n-2}$ for $n = 2, 3, \ldots$. The fact that $f/z \in M_{n-1,n-2}$ for $f \in M_{n,n-1}$ follows from L'Hôpital's rule.

The following theorem shows that a discontinuous derivation is completely determined by $D(z)$ on the square of the continuity ideal.

2.3 THEOREM. *Let $\mathcal{M}$ be a Banach $C^n[0,1]$-module with differential subspace W. Let D be a derivation of $C^n[0,1]$ into $\mathcal{M}$.*

(1) *If D is continuous then $D(f) = \gamma(f')D(z)$, $f \in C^n[0,1]$.*

(2) *If D is discontinuous with continuity ideal $\mathcal{T}(D) = M_{n,k-1}$ for some k, $1 \le k \le n$, then*

$$D(g) = \gamma\left(\frac{g'}{z^k}\right)[\rho(z^k)D(z)], \qquad g \in \mathcal{T}(D)^2.$$

The proof of (1) was given by Bade and Curtis in [**2**, Theorem 4.5].

PROOF OF (2). Let $g \in \mathcal{T}(D)^2$. Since $M_{n,k-1}^2 = z^k M_{n,k-1}$ we can write $g = z^k f$, where $f \in M_{n,k-1}$. Then $D(g) = \rho(z^k)D(f) + \rho(f)D(z^k)$. But $z^k \in \mathcal{T}(D)$ so that $\rho(z^k)D(\cdot)$ is continuous. By (1) we have

$$\rho(z^k)D(f) = \gamma(f')[\rho(z^k)D(z)].$$

Consider the Taylor expansion $f = \sum_{j=k}^n \delta_j(f)z^j + R_n f$, where $\delta_j(f) = f^{(j)}(0)/j!$ and $R_n f \in M_{n,n}$. Since $zM_{n,n-1}$ is dense in $M_{n,n}$, there exists $\{f_m\} \subset M_{n,n-1}$ such that $zf_m \Rightarrow R_n f$ in $C^n[0,1]$. By 2.1, $f_m \Rightarrow R_n f/z$ in $C^{n-1}[0,1]$ so that

$$f_m + \sum_{j=k}^n \delta_j(f)z^{j-1} \Rightarrow \frac{f}{z} \quad \text{in } C^{n-1}[0,1].$$

We have

$$\begin{aligned}
\rho(f)D(z^k) &= \lim_{m\to\infty} \rho\left(\sum_{j=k}^n \delta_j(f)z^j + zf_m\right) D(z^k) \\
&= \lim_{m\to\infty} \rho\left(\sum_{j=k}^n \delta_j(f)z^{j-1} + f_m\right) \rho(z)D(z^k) \\
&= \lim_{m\to\infty} \rho\left(\sum_{j=k}^n \delta_j(f)z^{j-1} + f_m\right) [k\rho(z^k)D(z)].
\end{aligned}$$

Since $\rho(z^k)D(\cdot)$ is continuous, $\rho(z^k)D(z) \in W$. By 1.2 we can replace ρ by γ so that

$$\begin{aligned}
\rho(f)D(z^k) &= \lim_{m\to\infty} \gamma\left(\sum_{j=k}^n \delta_j(f)z^{j-1} + f_m\right) [k\rho(z^k)D(z)] \\
&= \gamma\left(\sum_{j=k}^n \delta_j(f)z^{j-1} + \lim_{m\to\infty} f_m\right) [k\rho(z^k)D(z)] \\
&= \gamma\left(\frac{f}{2}\right) [k\rho(z^k)D(z)].
\end{aligned}$$

Now
$$\begin{aligned} D(g) &= \rho(z^k)D(f) + \rho(f)D(z^k) \\ &= \gamma(f')[\rho(z^k)D(z)] + \gamma\left(\frac{kf}{z}\right)[\rho(z^k)D(z)] \\ &= \gamma\left(\frac{z^k f' + kz^{k-1}f}{z^k}\right)[\rho(z^k)D(z)] \\ &= \gamma\left(\frac{g'}{z^k}\right)[\rho(z^k)D(z)]. \end{aligned}$$
Immediately following Remark 2.2 we have

2.4 COROLLARY. *Let D be a derivation of $C^n[0,1]$ into $\mathcal{M}$. If D is discontinuous with continuity ideal $\mathcal{T}(D) = M_{n,k-1}$ for some k, $1 \le k \le n$, then D is continuous on $\mathcal{T}(D)^2 = z^k M_{n,k-1}$ for the C^{n+k}-norm.*

The following result in [**2**] is a direct consequence of Theorem 2.3.

2.5 COROLLARY. *If D is a singular derivation of $C^n[0,1]$ then D vanishes also on $\mathcal{T}(D)^2$.*

2.6 COROLLARY. *Let D be a singular derivation of $C^n[0,1]$ with continuity ideal $\mathcal{T}(D) \subseteq M_{n,k-1}$ for some k, $1 \le k \le n$. Then the range of D is contained in the kernel of $\rho(z^k)$.*

PROOF. Let $f \in C^n[0,1]$. We write $f = \sum_{i=0}^{k-1} \delta_i(f)z^i + Rf$, where $Rf \in M_{n,k-1}$. Then
$$\rho(z^k)D(f) = \rho(z^k)D(Rf) = \rho(z^k)D(Rf) + \rho(Rf)D(z^k) = D(z^k Rf) = 0.$$

3. Application. We determine the structure of all derivations of $C^1[0,1]$ into $L_p(0,1)$, $1 \le p < \infty$, where the module multiplication is defined by
$$(*)\qquad (\rho(f)x)(t) = f(t)x(t) - \int_0^t f'(s)x(s)\,ds, \qquad f \in C^1[0,1],\ x \in L_p(0,1).$$
We need the characterization of the differential subspace for this module given by Bade and Curtis [**2**]. In the following if x is a function of bounded variation, we write $x(ds)$ and $v(x)(ds)$ for the measure corresponding to x and its total variation.

3.1 THEOREM. *Let $L_p(0,1)$, $1 \le p < \infty$, be given the $C^1[0,1]$-module operation defined by $(*)$. An element $x \in L_p(0,1)$ belongs to the differential subspace W if and only if*

(1) *x is of bounded variation on each interval $[0,t]$, $0 < t < 1$, and*
(2) $\int_0^1 v(x)([0,t])^p\,dt < \infty$.

Let x be in W; we can suppose x is right continuous. By Theorem 1.2, for any polynomial p we have
$$(\gamma(p)x)(t) = (\rho(p)x)(t) = p(t)x(t) - \int_0^t p'(s)x(s)\,ds.$$
An integration by parts yields
$$(\gamma(p)x)(t) = p(0)x(0) + \int_0^t p(s)x\,(ds),$$

so that

$$(\gamma(f)x)(t) = f(0)x(0) + \int_0^t f(s)x(ds), \qquad f \in C[0,1],\ x \in W.$$

Thus all continuous derivations $D\colon C^1[0,1] \to L_p(0,1)$ are of the form

$$D(f)(t) = (\gamma(f')D(z))(t) = f'(0)D(z)(0) + \int_0^t f'(s)D(z)(ds).$$

Now let $D\colon C^1[0,1] \to L_p(0,1)$ be a singular derivation with continuity ideal $\mathcal{T}(D) = M_{1,0}$, by Corolllary 2.6 $D(C^1[0,1]) \subseteq \ker\rho(z)$ which is one dimensional. Thus D is a point derivation.

We now turn to the structure of an arbitrary discontinuous derivation D of $C^1[0,1]$ into $L_p(0,1)$. We show D is the sum of a continuous linear map, a singular derivation and a discontinuous linear functional on $C^1[0,1]$. We need the following lemma.

3.2 LEMMA. *Let $D\colon C^1[0,1] \to L_p(0,1)$ be a discontinuous derivation with singularity set $F = \{0\}$. We can write $D = D_1 + D_2$ where D_1 is a discontinuous derivation and $D_2(z)$ is of bounded variation on $[a,1]$ for all $0 < a < 1$.*

PROOF. Since $\rho(z)D(\cdot)$ is a continuous derivation,

$$(\rho(z)D(f))(t) = tD(f)(t) - \int_0^t D(f)(s)\,ds \quad \text{is in } W.$$

It follows from Theorem 3.1 that $D(f)$ is a function of bounded variation on $[a,c]$, $0 < a < c < 1$, for all f in $C^1[0,1]$.

Fix b in $(0,1)$ and let $y = k_{[b,1]}D(z)$ where $k_{[b,1]}$ denotes the characteristic function on $[b,1]$. Then y is of bounded variation on $[0,c]$ for all $0 < c < 1$. Now

$$tD(z)(t) = (\rho(z)D(z))(t) + \int_0^t D(z)(s)(ds) \in W$$

so that $v(k_{[b,1]}zD(z))([0,t]) \in L_p(0,1)$. Since $1/t$ is boudned on $[b,1]$ it follows that $v(k_{[b,1]}D(z))([0,t])$ is also in $L_p(0,1)$. Thus $y \in W$. By [**2**, Proposition 4.2] there exists a continuous derivation $D_1\colon C^1[0,1] \to L_p(0,1)$ such that $D_1(z) = y$. Then $D_2 = D - D_1$ is a discontinuous derivation with $D_2(z) = k_{[0,b]}D(z)$ a function of bounded variation on $[a,1]$ for all $0 < a < 1$.

We are now in a position to describe discontinuous derivations from $C^1[0,1]$ into $L_1(0,1)$.

3.3 THEOREM. *Let $D\colon C^1[0,1] \to L_1(0,1)$ be a discontinuous derivation with singularity set $F = \{0\}$. By Lemma 3.2 we may assume that $D(z)$ is of bounded variation on $[0,1]$ for $0 < a < 1$. We can write*

$$D(f) = T(f) + S(f) + \alpha(f), \qquad f \in C^1[0,1],$$

where T is a continuous linear map from $C^1[0,1]$ into $L_1(0,1)$ which is completely determined by $D(z)$, S is a singular derivation and α is a discontinuous linear functional on $C^1[0,1]$.

PROOF. Let $\mu = \rho(z)D(z)$; then $\mu \in W$. We may assume that μ is right continuous so that $\mu(0) = 0$, moreover μ is of bounded variation on $[0,1]$ since

$D(z)$ is bounded variation on $[a,1]$. For f in $\mathcal{T}(D)^2 = M_{1,0}^2$ we have

$$D(f)(t) = \left(\gamma\left(\frac{f'}{z}\right)\mu\right)(t) = \int_0^t \frac{f'(s)}{s}\mu(ds)$$
$$\int_0^1 \frac{f'(s)}{s}\mu(ds) - \int_t^1 \frac{f'(s)}{s}\mu(ds).$$

The map $T\colon f \to -\int_t^1 (f'(s)/s)\mu(ds)$ defines a continuous linear map from $C^1[0,1]$ into $L_1(0,1)$ which is bounded by $\|f\|v(\mu)([0,1])$. The first integral defines a discontinuous linear functional defined on $M_{1,0}^2$ which can be extended to all of $C^1[0,1]$ via the Hahn Banach theorem. Let α be an extension of the first integral to $C^1[0,1]$ with $\alpha(1) = 0$ and $\alpha(z) = c$, any constant c. Define $\overline{D}\colon C^1[0,1] \to L_p(0,1)$ by $\overline{D}(f) = \alpha(f) - \int_0^1 (f'(s)/S)\mu(ds)$. Since the constant function is the eigenfunction at zero for $\rho(f)$, $f \in M_{1,0}$, $\overline{D}$ is a derivation. Since $\overline{D}(z^2) = D(z^2)$, $\overline{D}(z) - D(z) \in \ker\rho(z)$, thus we can choose c so that $\overline{D}(z) = D(z)$. Then $D - \overline{D}$ is a singular derivation S, and we have $D(f) = T(f) + S(f) + \alpha(f)$ as claimed.

4.3 THEOREM. *Let $D\colon C^1[0,1] \to L_p(0,1)$, $1 \le p < \infty$, be a discontinuous derivation with singularity set $F = \{0\}$. We may assume that $D(z)$ is of bounded variation on $[a,1]$ for $0 < a < 1$. We can write*

$$D(f) = T(f) + S(f) + \alpha(f), \qquad f \in C^1[0,1],$$

where $T\colon C^1[0,1] \to L_p(0,1)$ is a continuous linear map which is completely determined by $D(z)$, S is a singular derivation and α is a discontinuous linear functional on $C^1[0,1]$.

PROOF. Since $L_p(0,1) \subseteq L_1(0,1)$ for $p \ge 1$, we can consider D as a derivation from $C^1[0,1]$ into $L_1(0,1)$. By Theorem 3.3 we can write

$$D(f) = T(f) + S(f) + \alpha(f), \qquad f \in C^1[0,1],$$

so

$$T(f) = D(f) - S(f) - \alpha(f), \qquad f \in C^1[0,1].$$

Since all the terms on the left-hand side are in $L_p(0,1)$, $T(f) \in L_p(0,1)$ for all $f \in C^1[0,1]$. Let $y \in L_p(0,1)$ be in the separating space $\mathcal{S}(T)$ of T. There exists $f_m \Rightarrow 0$ in $C^1[0,1]$ and $T(f_m) \Rightarrow y$ in $L_p(0,1)$. By Theorem 3.3 T is a continuous linear map from $C^1[0,1]$ into $L_1(0,1)$ so that $T(f_m) \Rightarrow 0$ in $L_1(0,1)$. Thus $y = 0$ and we conclude that T is continuous. This completes the proof.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, CALIFORNIA STATE UNIVERSITY, LONG BEACH, CALIFORNIA 90840