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ANALYTIC AND DIFFERENTIABLE FUNCTIONS VANISHING ON AN ALGEBRAIC SET

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ABSTRACT. Let U be an open semi-algebraic subset of $\mathbf{R}^n$ and let X be a closed analytic subset of U which is also a semi-algebraic set (e.g., $U = \mathbf{R}^n$ and X is an algebraic subset of $\mathbf{R}^n$). It is proved that the ideal of analytic functions on U vanishing on X is finitely generated provided that the set X is coherent. The ideal of infinitely differentiable functions on U vanishing on X is finitely generated if and only if the set X is coherent.

1. The results. Let U be an open subset of $\mathbf{R}^n$. Denote by $\mathcal{O}(U)$ and $\mathcal{E}(U)$ the ring of analytic functions and C^∞ functions on U, respectively. Given a subset X of U, we define $I(X)$ to be the ideal of $\mathcal{O}(U)$ and $I_*(X)$ to be the ideal of $\mathcal{E}(U)$ of all functions vanishing on X. It would be interesting to know under what assumptions on X the ideals $I(X)$ and $I_*(X)$ are finitely generated (cf. [**1**, **4**]). For instance, it remains an open question whether the ideal $I(X)$ is finitely generated if X is an algebraic subset of $\mathbf{R}^n$ [**4**, p. 65].

We propose one result in this direction. Let $\mathcal{O}$ be the sheaf of germs of analytic functions on $\mathbf{R}^n$ and let $\mathcal{J}(X)$ be the subsheaf of ideals of $\mathcal{O}|U$ of germs vanishing on X. Recall that if X is a closed analytic subset of U, then X is said to be coherent if the sheaf $\mathcal{J}(X)$ is coherent. Denote by $\mathcal{E}$ the sheaf of germs of C^∞ functions on $\mathbf{R}^n$ and by $\mathcal{J}_*(X)$ the subsheaf of ideals of $\mathcal{E}|U$ of germs vanishing on X. We shall need the following characterization of coherent analytic sets.

PROPOSITION 1. *Let U be an open subset of $\mathbf{R}^n$ and let X be a closed analytic subset of U. Then X is coherent if and only if for each x in U the stalk $\mathcal{J}_*(X)_x$ is finitely generated over $\mathcal{E}_x$.*

Let us recall that a subset of $\mathbf{R}^n$ is said to be semi-algebraic if it belongs to the smallest family of subsets which contains sets of the form $\{x \in \mathbf{R}^n | p(x) > 0\}$, where $p\colon \mathbf{R}^n \to \mathbf{R}$ is a polynomial function, and which is closed under the Boolean operations of finite union, finite intersection and taking complements. Obviously, every algebraic subset of $\mathbf{R}^n$ is semi-algebraic.

THEOREM 2. *Let U be an open semi-algebraic subset of $\mathbf{R}^n$ and let X be a closed analytic subset of U which is also a semi-algebraic set. Then:*

(i) *The ideal $I(X)$ is finitely generated if the set X is coherent.*

(ii) *The ideal $I_*(X)$ is finitely generated if and only if the set X is coherent.*

REMARK 3. (i) If X is a closed coherent analytic subset of $\mathbf{R}^n$, then, in general, the ideal $I(X)$ is not finitely generated (cf. [**3**] for an idea of how to construct an example).

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(ii) Theorem 2 applies, in particular, in the case where $U = \mathbf{R}^n$ and X is an algebraic subset of $\mathbf{R}^n$. We wish to point out that in such a case the ideal $I(X)$ is not, in general, generated by polynomials. For example, the curve X in $\mathbf{R}^2$ given by the equation $x_2^2 - x_1^2(x_1 - 1) = 0$ is irreducible as an algebraic set and has an isolated point at the origin. It follows that the ideal $I(X)$ cannot be generated by polynomials. Clearly, X is coherent as an analytic set.

There is a conjecture (cf. [**4**, p. 65]) that the ideal $I(X)$ is always generated by so-called Nash functions if X is an algebraic subset of $\mathbf{R}^n$.

Let us mention that if Y is a complex algebraic subset of $\mathbf{C}^n$, then the ideal of holomorphic functions on $\mathbf{C}^n$ vanishing on Y is generated by polynomials and, hence, is finitely generated [**9**].

PROPOSITION 4. *Let X be an algebraic subset of $\mathbf{R}^n$. If the set of singular points of X is bounded, then the ideal $I(X)$ is finitely generated.*

We conclude this section by giving

EXAMPLE 5. Let $f(x_1, x_2, x_3) = x_3(x_1^2 + x_2^2) - x_1^3$ and let X be the set of zeros of f. One checks easily that the set of singular points of X is not bounded and X is not coherent as an analytic set. However, the ideal $I(X)$ is generated by f [**2**, Proposition 5.2(1)]. By Theorem 2, the ideal $I_*(X)$ is not finitely generated. (It is stated in [**2**, p. 85] that the ideal $I_*(X)$ is not generated by f but the proof of this fact is incorrect.)

2. The proofs. The proofs are based on the results of Malgrange, Merrien and Tougeron.

PROOF OF PROPOSITION 1. If the set X is coherent, then $\mathcal{J}_*(X) = \mathcal{J}(X)\mathcal{E}|U$ [**12**, p. 127, Theorem 4.2]. Since each stalk of $\mathcal{J}(X)$ is finitely generated so is each stalk of $\mathcal{J}_*(X)$.

Now assume that each stalk of $\mathcal{J}_*(X)$ is finitely generated. Let $\mathcal{F}_n$ be the $\mathbf{R}$-algebra of formal power series in n variables. Given a point x in $\mathbf{R}^n$, denote by $T_x\colon \mathcal{E}_x \to \mathcal{F}_n$ the homomorphism induced by the infinite Taylor expansion at x. It follows from [**7**, p. 90, Theorem 3.5] that $T_x(\mathcal{J}_*(X)_x) = T_x(\mathcal{J}(X)_x)\mathcal{F}_n$ for all x in U. Fix a point x in U and let $f_1, \ldots, f_k$ be analytic functions defined on a neighborhood U_x of x in U whose germs $f_{1x}, \cdots, f_{kx}$ at x generate $\mathcal{J}(X)_x$. Then $T_x(f_1), \ldots, T_x(f_k)$ generate $T_x(\mathcal{J}_*(X)_x)$. Clearly, $\mathcal{J}_*(X)_x$ is a closed ideal of $\mathcal{E}_x$ (cf. [**12**, pp. 98–99] for a definition of a closed ideal in $\mathcal{E}_x$). By [**5**, p. 48, Proposition 1], the germs $f_{1x}, \ldots, f_{kx}$ generate $\mathcal{J}_*(X)_x$. Since, by [**1**, Proposition 1.3], the sheaf $\mathcal{J}_*(X)$ is quasi-flasque, the germs $f_{1y}, \cdots, f_{ky}$ generate $\mathcal{J}_*(X)_y$ for all y in U_x provided that U_x is a sufficiently small neighborhood of x [**12**, p. 115, Proposition 6.4]. Therefore $T_y(f_1), \ldots, T_y(f_k)$ generate $T_y(\mathcal{J}(X)_y)\mathcal{F}_n$. By [**12**, p. 26, Proposition 8.2], the germs $f_{1y}, \ldots, f_{ky}$ generate $\mathcal{J}(X)_y$ for all y in U_x and, hence, X is a coherent set.

PROOF OF THEOREM 2. We claim that there exists a positive integer k such that the stalk $\mathcal{J}(X)_x$ can be generated by at most k elements of $\mathcal{O}_x$ for all x in U. Indeed, define $h\colon \mathbf{R}^n \to \mathbf{R}^n$ by $h(x) = x/(1 + \|x\|^2)^{1/2}$ for x in $\mathbf{R}^n$. Since the graph of h is semi-algebraic, the set $Y = h(X)$ is also semi-algebraic [**10**]. By

[**8**, Theorem 7.1], the subsheaf $\mathcal{J}(Y)$ of $\mathcal{O}$ over $\mathbf{R}^n$ is semifinite. Thus for each point x in $\mathbf{R}^n$ there exist a neighborhood U_x of x in $\mathbf{R}^n$ and a positive integer k_x such that $\mathcal{J}(Y)_y$ can be generated by at most k_x elements of $\mathcal{O}_y$ for all y in U_x. Since h diffeomorphically maps $\mathbf{R}^n$ onto the open unit ball in $\mathbf{R}^n$, the set Y is bounded and the claim follows.

If X is coherent, then the ideal $\Gamma(U, \mathcal{J}(X))$ of global sections of $\mathcal{J}(X)$ is finitely generated (cf. [**6**]). Since, by Theorem A of Cartan, $I(X) = \Gamma(U, \mathcal{J}(X))$, the proof of (i) is completed.

Again, if X is coherent, then $\mathcal{J}_*(X) = \mathcal{J}(X)\mathcal{E}|U$ [**12**, p. 127, Theorem 4.2]. It follows from [**12**, p. 214, Proposition 5.1] that the ideal $I_*(X) = \Gamma(U, \mathcal{J}_*(X))$ is finitely generated.

Now assume that the ideal $I_*(X)$ is finitely generated. Clearly, $\mathcal{J}_*(X)_x = I_*(X)\mathcal{E}_x$ for all x in U. By Proposition 1, the set X is coherent.

PROOF OF PROPOSITION 3. Let $\mathcal{J}$ be the sheaf of ideals associated with $I(X)$, i.e., $\mathcal{J}_x = I(X)\mathcal{O}_x$ for x in $\mathbf{R}^n$. By [**11**, Theorem 2], the sheaf $\mathcal{J}$ is coherent. Note that if x is a nonsingular point in x, then the ideal $\mathcal{J}_x$ is generated by polynomials vanishing on X. Since the set of singular points of X is bounded, there exists a positive integer k such that the ideal $\mathcal{J}_x$ can be generated by at most k elements of $\mathcal{O}_x$ for all x in $\mathbf{R}^n$. It follows that the ideal $I(X) = \Gamma(X, \mathcal{J})$ is finitely generated.

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